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THE PERCEPTION AND PROCESSING OF VISUAL NUMEROSITY

(Waarnemen en benoemen van visueel aantal)

Michiel van Oeffelen

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Proefschrift

ter verkrijging van de graad van doctor
in de sociale wetenschappen
aan de Katholieke Universiteit te Nijmegen
op gezag van de Rector Magnificus Prof. Dr. J. H. G. I. Giesbers
volgens het besluit van het College van Dekanen
in het openbaar te verdedigen
op donderdag 9 februari 1984
des namiddags te 4.00 uur

door

Michaël Pieter Johannes van Oeffelen

Geboren te Breda

Nijmegen 1984

krips repro meppel

Aan mijn ouders, Lia, en Thomas,

voor wie dit geheel meer is

dan de som zijner delen.

Woord vooraf

Het in dit proefschrift beschreven onderzoek is mogelijk gemaakt dankzij een subsidie van de Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (ZWO-projectnr: 15-32-05) en uitgevoerd binnen de vakgroep Ontwikkelingspsychologie van de Katholieke Universiteit te Nijmegen. Ik betuig mijn erkentelijkheid aan allen die op enigerlei wijze hebben bijgedragen tot de totstandkoming van dit proefschrift en met name Chris Bouwhuizen, Hans Jansen en Jos Wittebrood voor hun technische ondersteuning en Jan Sterenborg voor zijn computerprogrammatische hulp.

Nijmegen, december 1983 Michiel van Oeffelen

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Waarnemen en Benoemen van Visueel Aantal.

Inleiding.

Het zal eenieder als vanzelfsprekend overkomen dat we, vrijwel onmiddellijk en zonder nadenken, het aantal kunnen geven van een kleine verzameling van objecten, of dat nu kopjes, stoelen of stipjes zijn. Van grotere verzamelingen kan in de regel slechts tellenderwijs het aantal bepaald worden. De te volgen telstrategie is dan in belangrijke mate afhankelijk van de manier waarop de objecten zijn gerangschikt. Zo laat een verzameling, bestaande uit vijf groepjes van ieder vijf objecten, zich heel wat makkelijker tellen dan wanneer dezelfde objecten willekeurig zijn gerangschikt. Onder tijdsdruk evenwel, zal men slechts een schatting kunnen geven van het juiste aantal met als risico het maken van fouten. Hoe vanzelfsprekend ook, de perceptuele en cognitieve processen welke aan dergelijke gedragingen ten grondslag liggen, laten zich moeilijk in kaart brengen. Zie hier kort geschetst het probleemgebied waarop dit proefschrift zich begeeft.

Een eerste demonstratie omtrent de hoeveelheid objecten welke men in een oogopslag met zekerheid kan benoemen, werd bij wijze van gedachtenexperiment gerapporteerd door Sir William Hamilton (1865) in zijn Lectures on Metaphysics and Logic: "Als je een handvol knikkers op de grond gooit, dan zul je zien dat het moeilijk is om er meer dan 6 à 7 in een oogopslag en zonder verwarring te overzien; maar groepeer je ze in tweeën, drieën, of zelfs in vijven, dan kun je net zoveel groepjes overzien als dat je eenheden kunt overzien, want de aandachtszin beschouwt deze groepen slechts als eenheden" (Hamilton, 1865, p.254). Jevons (1871) was de eerste die Hamilton's gedachtenexperiment aan empirische toetsing onderwierp. Hij plaatste een klein wit schoteltje te midden van een grotere, zwarte bak en gooide een handvol donkere bonen op zo dat een gedeelte ervan op het witte schoteltje viel. Hij schatte "zonder de minste aarzeling" het aantal bonen welke in het witte schoteltje terecht was gekomen, noteerde dit, waarna hij het werkelijke aantal telde. Na zo'n duizendtal metingen verbaasde Jevons zich erover dat hij zelfs vergissingen maakte bij 5 bonen, maar vooral ook over het stijgende aantal fouten met toenemend aantal bonen op het schoteltje. Een verklaring voor zijn bevindingen kon hij niet geven maar vermoedde wel dat de relatie tussen schatting en aantal gevangen kon worden onder een algemene regel.

Mede door de ontwikkeling van verfijnde instrumenten als de tachistoscoop en reactietijdsapparatuur richtte de belangstelling zich nadien vooral op het meten van de aandachtsspanne en op factoren welke het bereik ervan beïnvloeden. Reactietijden (RT's) als functie van aantal werden verzameld in taken waar het erom ging zo snel mogelijk het juiste aantal objecten binnen een aangeboden stimulus te bepalen. Stimuli bestonden meestal uit stipfiguren waarbinnen de rangschikking der stippen gerandomiseerd was. Kenmerkend voor het functioneel verband tussen RT en aantal was het optreden van een discontinuïteit waarbeneden RT's klein waren en nauwelijks van elkaar verschilden, en waarboven RT's snel toenamen met toenemend aantal stippen. Bij tachistoscopische proeven was de stimulusaanbieding erg kort (ongeveer 100 msec.). Analyse van correcte responsies gegeven op een aangeboden aantal stippen resulteerde in een functioneel verband tussen het percentage correcte responsies en het aantal stippen. Dalend als functie van aantal doorsneed de curve een kritisch nivo (gewoonlijk het 50-procentsnivo) op een plaats welke overeenkwam met de plaats waar bovengenoemde discontinuïteit optrad. Deze plaats werd algemeen aanvaard als de bovengrens van de aandachtsspanne. Het zou te ver gaan hier alle waarden te vermelden welke door diverse onderzoekers werden gerapporteerd. We volstaan met de opmerking dat het gros der bevindingen een bovengrens aantoonde gelegen bij aantal 6 à 7 met een enkele uitschieter naar beneden (aantal=4, Atkinson, et.al., 1976) en naar boven (aantal=8; Woodworth & Schlosberg, 1954). De verschillen zijn echter vaak te herleiden tot verschillen in procedure, stimulusmateriaal, en proefpersonenbestand (voor een uitvoerig historisch overzicht zijn verschillende bronnen beschikbaar waaronder te noemen: Atkinson, et.al., 1976, pag. 327-334; Beckwith & Restle, 1966, pag. 437-444; Mandler & Shebo, 1982, pag. 1-22; Woodworth & Schlosberg, 1954, pag. 140-170).

Operationalisering van aantal als variabele in het experimentele onderzoek naar het fenomeen der aandachtsspanne had een onderkenning van aantalspecifieke strategieën tot gevolg. De term ´subitizing´ werd door Kaufman et.al. (1949) geïntroduceerd voor het zeer snelle en accurate benoemingsgedrag op aantallen binnen de aandachtsspanne. Als het om aantal ging sprak men dan ook niet meer over aandachtsspanne maar over subiteerspanne. Tellen heette simpelweg de benoemingsstrategie voor het exact kwantificeren van aantallen buiten deze spanne, schatten was de voor de hand liggende term voor de wel snelle maar overigens onnauwkeurige kwantificering van die aantallen.

De verschuiving van de aanvankelijk filosofische interesse voor aandachtsonderzoek naar empirisch aantalbenoemingsonderzoek leverde aldus een kwantitatief beschrijvingskader op waarbinnen aan de diverse benoemingsstrategieën een duidelijke plaats kon worden toebedeeld. Vooralsnog echter bleef het op z´n minst onduidelijk welke ´processen´ aan die strategieën ten grondslag liggen. Het kan in dit verband geen kwaad om nog eens stil te staan bij wat vroegere filosofen over het mogelijke verband tussen aantalbenoeming en aandacht naar voren hebben gebracht.

Reeds lang voordat problemen over aandacht binnen het bereik van de empirische psychologie kwamen (zie Wundt: Grundzüge der Physiologischer Psychologie, 1903) was door filosofen reeds diepgaand ingegaan op de vraag of een mens een dan wel meerdere dingen terzefdertijd kon opnemen. Met ´terzelfdertijd´ werd dan bedoeld ´op een en hetzelfde ondeelbare moment´. De visie van Aristoteles is hier het vermelden waard. In een van zijn minder bekende geschriften, de Parva Naturalia welke ten dele handelt over de zintuiglijke waarneming, stelde hij dat het "onmogelijk is om twee dingen tegelijkertijd waar te nemen, tenzij ze gecombineerd zijn, want de combinatie van beide <u>is</u> één object. De geldingsdrang van ieder afzonderlijk object binnen de combinatie is dan noodzakelijk kleiner dan wanneer deze ook afzonderlijk zouden worden beschouwd." (Aristoteles: Parva Naturalia, On the soul, pag. 447b 15-25) De gedachtengang van Aristoteles werd later door Hamilton (1865) concreet weergegeven middels de postulering van een soort van wet over de kennis. De wet, welke Hamilton de ´Law of Limitation´ noemde, zegt dat de intensie van onze kennis omgekeerd evenredig is met zijn extensie, met andere woorden, hoe minder objecten we onmiddellijk beschouwen, des te helderder en meer distinct zal onze kennis erover zijn. Deze uit de filosofie over het zintuiglijk waarnemen afgeleide ´wet´ was in feite de kiemcel voor de theorievorming die in het eerste hoofdstuk van dit proefschrift ontwikkeld en getoetst wordt.

Hoofdstuk 1 geeft een beschrijving van een waarschijnlijkheidsmodel voor de onmiddellijke waarneming van visueel aantal. In het model wordt de kwantiteit ´aantal´ beschouwd als een fysische kwaliteit. Bekend was dat voor fysische variabelen als auditieve frequentie en lichtintensiteit Thurstone´s wet van de vergelijkende oordelen geldt. Toegesneden op de kwaliteit ´aantal´ van een stippenfiguur wordt verondersteld dat bij presentatie ervan een discriminatieproces wordt getriggerd waarbij de externe stimulusgrootheid (aantal n) getransformeerd wordt naar een waarde op een intern (psychologisch) continuum. Woodworth & Schlosberg (1954) merkten op dat voor aantal de transformatie niet lineair is: Het subjectieve verschil tussen aantal 2 en 3 is groter dan tussen 3 en 4, en dat weer groter dan het verschil tussen 4 en 5, enzovoort. De veronderstelling dat de interne schaal logarithmisch verloopt (wet van Fechner; zie voor Weber-Fechner theorie: Guilford, 1954) ten opzichte van de externe lineaire getallenschaal komt aan deze visie tegemoet. Door ruisfactoren, o.a. in de transmissie van informatie, laat zo'n interne representatie van aantal zich beschrijven door een random variabele. Het is dan verder een kwestie van elementaire statistiek om vervolgens te laten zien dat visuele discriminatie restricties oplegt aan aantalbenoeming. In Hoofdstuk 1 zal onder gebruikmaking van deze vooroverwegingen als hoofdstelling getoetst worden dat subiteren verklaard kan worden als zijnde het gevolg van een bovendrempelig onderscheid tussen kleine naburige aantallen. Tevens zal aangetoond worden dat de bovengrens van de subiteerspanne gemarkeerd wordt door een Weber-drempel voor visueel aantal, in overeenstemming met hetgeen daaromtrent door Averbach (1963) reeds werd geopperd. Naast de presentatie van een model voor de onmiddellijke waarneming van visueel aantal wordt in Hoofdstuk 1 verder verslag gedaan van een tweetal experimenten, - een reactietijdsexperiment en een drempelexperiment-, waarin het model op zijn waarde wordt getoetst.

Uiteraard is het zo dat een verzameling objecten, behalve door het aantal objecten, gekenmerkt wordt door de ligging der objecten, verder rangschikking of configuratie genaamd. Modellen over strategieën, te hanteren tijdens aantalbenoemingstaken, dienen derhalve behalve aan het aantal objecten tevens gerelateerd te worden aan configurationele eigenschappen van een stippenfiguur. We herinneren in dit verband aan het voorbeeld van een verzameling bestaande uit 25 stippen welke, in groepjes van 5 opgedeeld, een veel zuiniger benoemingsstrategie toestond dan wanneer de verzameling stippen onregelmatig verdeeld zou zijn. Alvorens de invloed van figuur-aantal interactie op aantalbenoeming te beschouwen dienen we ons eerst de vraag te stellen hoe groeperingen in verzamelingen worden waargenomen. Immers, binnen stipfiguren is er geen enkele dwingende noodzaak tot het vormen van groeperingen. Waarneming ervan is vooralsnog subjectief. Als leidraad voor subjectief groeperen zijn de Gestaltregels (Koffka, 1935; Köhler, 1947; zie voor uitvoerige bespreking: Metzger, 1953; Wertheimer, 1923) algemeen aanvaard maar bij herhaling is opgemerkt (Zusne, 1970; Utal, 1975) dat deze geen kwantitatieve beschrijving geven. Hun geldingskracht wordt slechts ontleend aan demonstratie ervan. Hoofdstuk 2 biedt wellicht de eerste psychologisch relevante formalisering van een van de belangrijkste pregnantie-determinerende Gestaltregels, namelijk die der nabijheid. Het hier gepresenteerde model (CODE) geeft een voorschrift hoe, op basis van relatieve nabijheid, groeperingen binnen stipfiguren worden ontdekt overeenkomstig een van de trekken van een menselijke waarnemer. De psychologische validiteit en inzetbaarheid ervan zullen verder uitvoerig worden bediscussiëerd.

Experimentele steun voor CODE wordt vervolgens gezocht in een drempelexperiment waarbij door CODE gecontroleerde stip figuren kortstondig worden aangeboden. De taak is èen aantal-discriminatie-taak waar nu niet naar het aantal stippen maar naar het aantal waargenomen groepjes gevraagd wordt. In deze vorm gegoten vertoont de taak sterke overeenkomst met Hamilton's demonstratie zoals die eerder in de inleiding is besproken. De experimentele uitwerking vindt zijn beslag in het eerste deel van Hoofdstuk 3.

Het ontbreken van een kwantitatieve beschrijving van figuureigenschappen binnen een stippenverzameling is er waarschijnlijk de oorzaak van dat de invloed ervan op een toch alledaags gedrag als tellen nauwelijks systematisch is onderzocht. Diverse onderzoekers (Atkinson, et.al., 1976; Bourdon, 1908; Fernberger, 1921; Freeman, 1912) rapporteerden weliswaar een variatie in de locatie van de bovengrens van subiteren als gevolg van patrooninvloeden. Voor aantallen duidelijk boven die grens is eigenlijk maar één studie, die van Beckwith & Restle (1966) later gerepliceerd door Aoki (1977), bekend waar de invloed van rangschikking der punten op het telgedrag werd onderzocht. In een onderzoek ter verificatie van een model waar één-voor-één tellen als strategie centraal stond gebruikten zij vier verschillende varianten voor een zelfde aantal te weten lineair, cirkelvormig, rechthoekig, en random. Een-voor-een tellen bleek evenwel een te simpele voorstelling van de gehanteerde strategie. Eerder leek het erop dat proefpersonen een stippenpatroon opdeelden in kleinere, subiteerbare groepjes, de aantallen aldus verzamelend om deze ofwel toe te voegen aan een lopend totaal ofwel op te tellen aan het eind van het proces. De suggestie, geopperd door Beckwith & Restle (1966) vormt het vertrekpunt voor een in het tweede deel van Hoofdstuk 3 te presenteren model welke de invloed van figuur-aantal interactie op de benoeming van grotertallige stipfiguren beschrijft. Verder wordt er verslag gedaan van een reactietijdsexperiment waarin de voorspelbaarheid van het model aangaande latenties als functie van zowel aantal als groepeerbaarheid der stippen wordt getoetst.

Latenties, verzameld in reactietijdsexperimenten op grotertallige stipfiguren kunnen slechts indirect iets zeggen over de aard van de gevolgde benoemingsstrategie. De beschikbaarheid van apparatuur voor de meting van oogbewegingen bood de mogelijkheid om tijdens de taak het benoemingsgedrag rechtstreeks te volgen. Een voorwaarde is evenwel dat de oogpositie en zijn verandering over tijd in belangrijke mate is gerelateerd aan de gevolgde strategie. Met betrekking tot stimuluseigenaardigheden ligt het voor de hand te veronderstellen dat het oog gedurende de taak gericht is op informatierijke gebieden binnen het stippenpatroon. Spatiële informatie afleidbaar uit oogbewegingsregistraties kan dan wellicht patrooneigenaardigheden die het benoemingsgedrag beïnvloeden aanduiden. Daarenboven kan de temporele informatie leiden tot een scherper begrip omtrent deelprocessen welke aan een gevolgde strategie ten grondslag liggen. Vanwege zijn elegantie verdient de feitelijke meetmethode hier enige aandacht. Die komt erop neer dat een klein infra-rood bronnetje, gericht op het oog, een tweetal reflecties hierop veroorzaakt, -een pupilreflectie en een hoornvliesreflectie-, welke een verschilrelatie opleveren die eenduidig de oogbolstand, en daarmee de plaats van fixatie vastlegt. De relatie is onafhankelijk van de positie van het hoofd zodat hoofdbewegingen in beperkte mate zijn toegestaan. Praktisch voordeel daarbij is dat het hoofd niet behoeft te worden ingesnoerd zoals vaak bij andere methoden het geval is. De apparatuur bezit een microprocessor welke, gegeven de verschilrelatie van de beide reflecties de coördinaten van de oogfixatie bepaalt met de regelmaat van het videosignaal (50hz). Een calibratieprocedure is hier noodzakelijk om de posities te transformeren naar posities in het systeem van het stimulusveld. Een uitgebreid computerprogrammabestand verzorgt de dataverwerking met als resultaat trajecten van oogbewegingen uitgedrukt in duren en posities van fixaties, oogsprongen en perioden van oogknipperingen. De beschreven methode wordt ten volle ingezet bij een tweetal reactietijdsexperimenten waarvan verslag gedaan wordt in respectievelijk Hoofdstuk 4 en Hoofdstuk 5.

Hoofdstuk 4 behelst een verslag van een soortgelijk experiment als besproken in Hoofdstuk 3. Tijdens het benoemen door volwassen proefpersonen van grotertallige stimuli worden nu echter ook oogbewegingen gemeten. De extra informatie die daarmee gewonnen wordt zal besproken worden met betrekking tot het in Hoofdstuk 3 gepresenteerde model voor benoeming van bovensubiteerbare verzamelingen. Er worden echter ook kritische kanttekeningen gemaakt ten aanzien van de genoemde meetmethode. Vooruitlopend hierop is het bij voorbeeld mogelijk dat het centrum van het gezichtsveld niet persé hoeft samen te vallen met het centrum van het aandachtsveld. Een gevolg daarvan is dat belangrijke informatie parafoveaal kan worden ingewonnen zonder dat de oogpositie daar een aanduiding voor geeft.

Het vijfde en laatste Hoofdstuk van dit proefschrift plaatst het huidige benoemingsonderzoek in een ontwikkelingsperspectief. Piaget's studie over aantal (1952) betreft het belangrijke probleem van de ontwikkeling van conservatie van aantal. Zijn experimentele bevindingen wezen erop dat kinderen van 5 jaar of jonger niet inzien dat aantal (in Piaget's experimenten: knikkers, bloemetjes) invariant blijft onder spatiële transformaties. De oriëntatie van Piaget en van talloze van zijn navolgers heeft er toe geleid dat de belangstelling voor hoe kinderen in feite tellen tot dusverre gering is gebleven. Opgemerkt zij in dit verband ook dat de sinds de 60-er jaren sterk in de belangstelling gekomen Russische psychologen als Gal'perin (1969) en Davydov (1975) weliswaar het rekenen door kinderen als kernthema van onderzoek hebben ontwikkeld maar bij deze onderzoekers en hun navolgers staat niet centraal een beschrijving van de achterliggende cognitieve processen doch veeleer de optimalisering van rekendidactiek. Experimenteel-cognitief onderzoek naar aantalbenoeming bij jonge kinderen is betrekkelijk laat en dan nog slechts in weinig studies aan de orde gesteld (Beckwith & Restle, 1966; Fodor, 1972; Fuson et.al., 1982; Gelman & Gallistel, 1978; Smitsman, 1982). In een ontwikkelingsstudie benadrukten Gelman & Gallistel (1978) het belang om zowel de toeëigening van een representatie van aantal als de ontwikkeling van rekenvaardigheid te onderzoeken. Hoe kinderen binnen een leeftijdsbereik van 2 tot 8 jaar zich de telwoorden van 1 tot 100 aanleren werd uitvoerig beschreven door Fuson et.al. (1982). Volgens deze onderzoekers worden telwoorden aanvankelijk geleerd als onderdeel van een zich steeds uitbreidende reeks waarna geleidelijk de telwoorden als zelfstandig begrip uit de reeks gelicht worden om ze te gebruiken voor rekenkundige doeleinden. Onderzoek van Smitsman (1982) naar proportioneel schatten wees uit dat kinderen hun oordeel over welke verzameling meer elementen bevat voornamelijk baseren op inspectie van kleinere subiteerbare groepjes van elementen van de verzameling. Dit proces, door Smitsman 'sampling by groups' genoemd, suggereert dat een configuratie, afhankelijk van diens presentatietijd, herhaaldelijk geïnspecteerd wordt waarbij de combinatie van deeluitkomsten van meerdere van die inspecties bepalend is voor de uiteindelijke schatting. Systematische observaties (Beckwith en Restle, 1966; Gelman en Gallistel, 1978) tonen aan dat de meeste kinderen van rond vijf jaar bij het tellen van een verzameling de objecten aanwijzen en op een 'dreunerige' manier de aantallen aftellen. Beckwith en Restle (1966) merkten evenwel op dat kinderen net als volwassenen grote gevoeligheid vertonen voor de organisatie ofwel rangschikking van een verzameling, hetgeen doet vermoeden dat ook het telgedrag bij kinderen door groeperingseffecten beïnvloed kan worden. Met andere woorden, het zou wel eens kunnen zijn dat kinderen niet van volwassenen verschillen in de aard der gekozen benoemingsstrategie maar in de geringere efficiëntie, snelheid en reikwijdte ervan. Dit vermoeden is met name naar voren gebracht door Fodor (1977). In Hoofdstuk 5 wordt een verslag uitgebracht van een analyse van oogbewegingstrajecten van kleuters zoals geregistreerd tijdens het uitvoeren van eenvoudige teltaken. Er zal evidentie voor de juistheid van Fodor's veronderstelling worden aangevoerd uit informatie van zowel oogbewegingen als de gelijktijdig daarmee vergaarde chronometrische aspecten van de taakuitvoering.

De dissertatie wordt afgesloten met een slotbeschouwing waarin de belangrijkste conclusies nog eens op een rijtje worden gezet. Bovendien worden er, op basis van het hier gerapporteerde, openingen aangereikt naar tenminste een tweetal probleemgebieden. Het ene gebied omsluit een aantal vragen met betrekking tot het onder- en overschatten van rangschikkingen in samenhang met spatiële illusies. Een tweede terrein betreft een verrijking van CODE zodanig dat het zowel nabijheid als continuering aan kan. Het is van belang dat het aanvankelijk als doel op zich gestelde experimentele onderzoek naar het waarnemen en benoemen van visueel aantal als middel wordt ingezet ter ontsluiting van beide onderzoeksgebieden.

Referenties.

- Aoki, T. On the counting process of patterned dots. Tohuku Psychologica Folia, 1977, 36 (1-4), 15-22.

- Aristoteles. Parva Naturalia, On the Soul. Trans. W.S. Hett. London: William Heinemann LTD, 1957.

- Atkinson, J. Cambell, F.W., & Francis, M.R. The magic number 4±0: A new look at visual numerosity judgements. Perception, 1976, 5, 327-334.

- Atkinson, J., Cambell, F.W., & Francis, M.R. The dependence of the visual numerosity limit on orientation, colour, and grouping in the stimulus. Perception, 1976, 5, 335-342.

- Averbach, E. The span of apprehension as a function of exposure duration. Journal of Verbal Learning and Verbal Behavior, 1963, 2, 60-64.

- Beckwith, M., & Restle, F. Process of enumeration. Psychological Review, 1966, 73, 437-444.

- Bourdon, B. Sur le temps nécessaire pour nominer les nombres. Revue Philosophique de la France et de l´Etranger, 1908, 33, 426-431.

- Davydov, V.V. The psychological characteristics of the ´prenumerical´ period of mathematics instruction. In L.D. Steffe (Eds.), Soviet studies in the psychology of learning and teaching mathematics Chicago: The University of Chicago, 1975.

- Fernberger, S.W. A preliminary study of the range of visual apprehension. American Journal of Psychology, 1921, 32, 121-133.

- Freeman, F.N. Grouped objects as concrete basis for the number idea. Elementary School Teacher, 1912, 32, 306-314.

- Fuson, K.C., Richards, J. and Briars, D.J. The acquisition and elaboration of the number word sequence. In C.J. Brainerd (Ed.), Children's logical and mathematical cognition. Berlin: Springer-Verlag, 1982.

- Gal'perin, P.Y. and Georgiev, L.S. The formation of elementary mathematical notions. In J. Kilpatrick and I. Wirszup (Eds.), Soviet studies in the psychology of learning and teaching mathematics (vol.1). Chicago: The University of Chicago Press 1969.

- Gelman, R. and Gallistel, C.R. The child's understanding of number. Cambridge: Harvard University Press, 1978.

- Guilford, J.P. Psychometric Methods. New-york: McGraw-Hill Book Company, 1954.

- Hamilton, W. Lectures on Metaphysics and Logic. London: Blackwood, 1865.

- Jevons, W.S. The power of numerical discrimination. Nature, 1871, 3, 281-282.

- Kaufman, E.L., Lord, M.W., Reese, T.W., & Volkman, J. The discrimination of visual number. American Journal of Psychology, 1949, 62, 498-525.

- Koffka, K. Principles of Gestalt Psychology. London: Routledge and Kegan

Paul, 1962.

- Köhler, W. Gestalt Psychology. New York: Mentor Books, 1947.

- Mandler, G. and Shebo, B.J. Subitizing: An analysis of its component processes. Journal of Experimental Psychology: General, 1982, 111, (1), 1-22.

- Metzger, W. Gesetze des Sehens. Frankfurt-M: Verlag-Waldemar Kramer, 1953.

- Piaget, J. The Child's Conception of Number. London: Routledge & Kegan Paul, 1952.

- Thurstone, L.L. A law of comparitive judgment. Psychological Review, 1972, 34, 273-286.

- Thurstone, L.L. Fechner's law and the method of equal appearing intervals. Journal of Experimental Psychology, 1929, 12, 214-224.

- Utal, W.R. An Autocorrelation Theory of Form Detection. New York: John Wiley and Sons, 1975.

- Wertheimer, M. Untersuchungen zur Lehre von der Gestalt I. Psychologische Forschung, 1923, 4, 301-350.

- Woodworth, R.S., & Schlosberg, H. Experimental Psychology. New York: Holt, Rhinehart, and Winston, 1954.

- Wundt, W. Grundzüge der physiologischen Psychologie. Leipzig: Engelmann, 1903.

- Zusne, L. Visual Perception of Form. New York: Academic Press, 1970.

Perception & Psychophysics
1982 32 (2) 163 170

A probabilistic model for the discrimination of visual number

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This paper proposes a probabilistic model of how humans identify the number of dots within a briefly presented visual display The model is an application of Thurstone s law of comparative judgment, and it is assumed that the internal representation of numerosity consists of log-spaced random variables The discrimination between any two different numerosities is consequently described as a function of max/min, where max and min are the larger and smaller numbers, respectively The model was tested in two experiments in which the Weber fraction for numerosity, corresponding with the critical ratio of max and min, was found to have the value of 162 It was concluded that the classical span of subitizing numerosity is but a special case of the span of discrimination

If one is asked to estimate under time pressure the number (n) of dots in a display, response accuracy appears to be high when the number of dots does not exceed about 6, but decreases rapidly for larger values of n Under self-paced task conditions, response latencies show a similar pattern Latencies are very fast for small numbers and slow down considerably for n larger than about 6. Taves (1941) studied judgments of numerousness and stated that two mechanisms were involved—one used for up to 7 dots, the other for larger fields For the first mentioned discriminatory process, Kaufman, Lord, Reese, and Volkman (1949) proposed the term "subitizing "

Bourdon (1908) suggested that small numbers of dots are apprehended by immediate cognition: "1, 2, 3, 4 are thus sensations just like green, red, round, or square, the quality of twoness of a group of objects is essentially perceived in the same way as the quality red or round" (p 430, translation by the authors) There are certainly reasons for doubting the supposition that subitizing is some sort of purely holistic information processing as Bourdon suggested One such reason is the frequently reported finding that latencies also tend to increase within the subitizing range. Woodworth and Schlosberg (1954) noticed that the differences in question fit well with what is known about choice reaction times: "The bigger a difference, the more quickly it is perceived; and the (relative) difference between 1 and 2 is greater than that between 2 and 3, and so on up the scale In identifying 5 dots you have to distinguish this number from 4 and 6, identifying 2 dots, you need only make the easier discrimination between 2 dots and 1 and 3" (p 98) This hypothesis was considered further by Averbach (1963) in a study on the span of visual apprehension He suggested that the discrimination of visual number may depend on the ratio of the difference in magnitude between two numbers and the magnitude of an actually presented number. If that ratio is at least as large as the (hypothetical) Weber fraction for visual number, then, according to Averbach, judgmental accuracy and speed would have the characteristics attributed to subitizing number

Moyer and Landauer (1967) measured reaction time for deciding which of two simultaneously presented digits was larger and observed that reaction time decreased monotonically as the difference between the two numbers increased In interpreting their data, they suggested that the displayed numerals were converted to analogue magnitudes, and a comparison was then made between the magnitudes in much the same way that comparisons are made between physical stimuli The "internal magnitude" should then be a nonlinear compressed function of the magnitude of the digit In such compressive spacing, 8 and 9 would be closer together on the internal scale than 7 and 8, 7 and 8 closer together than 6 and 7, and so on

The present paper proposes a simple theory for the discrimination of visual number The theory essentially is an application of Thurstone's law of comparative judgment (Thurstone, 1927; Torgerson, 1958).

Model of Number Discrimination

We consider stimuli consisting of sets of n dots. All physical aspects of the dots, such as brightness and area, are kept constant throughout Moreover, ar-

Portions of this paper benefited from discussions with Prof Dr E Eijkman The authors wish to thank an anonymous reviewer This research was supported by a grant from The Netherlands Organization for the Advancement of Pure Research (Zuiver Wetenschappelijk Onderzoek) The authors' mailing address is Psychology Department, KUN, Montessorilaan 3, 6500 HE Nijmegen, The Netherlands

rangement of dots within a stimulus is considered to be random. Hence, a stimulus is described by the number, n, of dots only.

We suppose that the discriminability of such stimuli obeys Thurstone's law of comparative judgment (Thurstone, 1927, Torgerson, 1958). Briefly summarized, this law states that a (presented) stimulus triggers a discriminal process in which the external stimulus value n is transformed into some value on an internal psychological continuum. Because of internal noise factors' acting on the transmission of number information, the internal representation of number is described by a random variable.

$$n \mid \rightarrow \frac{1}{\sigma_n\sqrt{2\pi}}\exp-\frac{(q_n-x)^2}{2\sigma^2}. \quad (1)$$

Now, supposing that the subject is briefly presented a display containing n dots, we can specify the probability that the subject will report the correct number n as follows

$$P_I(n \mid n) = \frac{1}{\sigma_n\sqrt{2\pi}}\int_{C(n,n-1)}^{C(n,n+1)}\exp-\frac{(q_n-x)^2}{2\sigma_n^2}\,dx. \quad (2)$$

The subscript I in the conditional probability $P_I(n \mid n)$ indicates that all integer values are alternatives The lower and upper integral limens, $C(n,n-1)$ and $C(n;n+1)$, denote the category bounds for correct responses to n.

It is not important to know, a priori, the precise form of the psychological function q_n in Equation 2. Any positive monotonic function could be reasonable. Here, it is assumed that q_n is a logarithmic function. This choice satisfies Fechner's law as it was incorporated in Thurstone's (1929) theory of comparative judgments of visual numerosity It then follows that the category bounds $C(n;n-1)$ and $C(n;n+1)$ are positioned exactly halfway between the internal representations of n and $n-1$ for $C(n,n-1)$ and n and $n+1$ for $C(n;n+1)$ (Parducci, 1963) Consequently, Equation 2 can be rewritten as.

$$P_I(n \mid n) = \frac{1}{\sigma_n\sqrt{2\pi}}\int_{\frac{1}{2}[\ln(n)+\ln(n-1)]}^{\frac{1}{2}[\ln(n)+\ln(n+1)]}\exp-\frac{[\ln(n)-x]^2}{2\sigma_n^2}\,dx \quad (3)$$

By substituting y for

$$\frac{\ln(n)-x}{\sigma_n},$$

it follows directly that.

$$P_I(n \mid n) = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2\sigma_n}\ln\left(\frac{n-1}{n}\right)}^{\frac{1}{2\sigma_n}\ln\left(\frac{n+1}{n}\right)}\exp\left(-\frac{y^2}{2}\right)dy \quad (4)$$

Note that

$$\frac{1}{2\sigma_n}\ln\left(\frac{n+1}{n}\right) \text{ and } \frac{1}{2\sigma_n}\ln\left(\frac{n-1}{n}\right)$$

really are z scores, because they are the integral limits of a standardized normal distribution For reasons of parsimony, we assume the dispersion value σ_n to be constant for all n, $\sigma_n \equiv \sigma$, that is, σ_n is only affected by internal noise due to momentary fluctuations of the organism regardless of the magnitude of n To get some idea of what value the dispersion σ of the internal representation should have, we can estimate σ from the frequently reported finding that $P_I(7 \mid 7) = .5$ (see, for instance, Averbach, 1963, Hunter & Sigler, 1940). From

$$P_I(7 \mid 7) = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2\sigma}\ln\left(\frac{6}{7}\right)}^{\frac{1}{2\sigma}\ln\left(\frac{8}{7}\right)}\exp\left(-\frac{y^2}{2}\right)dy = .5, \quad (5)$$

we then find $\sigma = .1080$. With this value for σ, we can determine $P_I(n \mid n)$ for all n Figure 1 shows these conditional probabilities as a function of n.

In a situation where n is briefly presented, and the choice is between two alternatives, n and m, the probability that a subject will respond with n is:

$$P_{n,m}(n \mid n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{2\sigma}\ln\left(\frac{m}{n}\right)}\exp\left(-\frac{y^2}{2}\right)dy, \quad (6)$$

for $m > n$

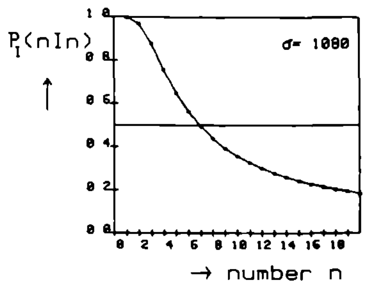

Figure 1 $P_I(n \mid n)$ plotted as a function of n, for $\sigma = .1080$

and

$$P_{n,m}(n|n) = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2\sigma}\ln\left(\frac{m}{n}\right)}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy, \quad (7)$$

for m < n

Defining "max" for the maximum value of n and m, and "min" for the minimum value of both, we can then write the general expression for the conditional probability that a response n will be given when n is presented in the presence of an alternative m:

$$P_{n,m}(n \mid n) = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2\sigma}\ln\frac{min}{max}}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy. \quad (8)$$

$P_{n,m}(n \mid n)$ is illustrated in Figure 2 for $n = 12$, alternative $m = 10$, and $\sigma = .1080$. $P_{12,10}(12 \mid 12)$ is the surface under the normal distribution function

$$f(12) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{[\ln(12) - x]^2}{2\sigma^2}$$

extending from ½ln(10 · 12) to infinity. The Weber threshold is defined usually as the 50% correct discrimination between two stimuli. Operationally, this difference limen is a stimulus difference that is noticed 75% of the time. From $P_{n,m}(n \mid n) = .75$, we can derive $(1/2\sigma)\ln(max/min)$, from which the Weber fraction for visual number, $W_b = (max - min)/min$ is easily calculated. It should be noted that the limen to the left (smaller numbers) of an actually presented number is closer to that number than the limen to the right of it (larger numbers). More specifically, defining n_l as that number, smaller than n, that can be discriminated from n 50% of the time, we get for the Weber fraction:

$$Wb = \frac{n - n_l}{n_l}. \quad (9)$$

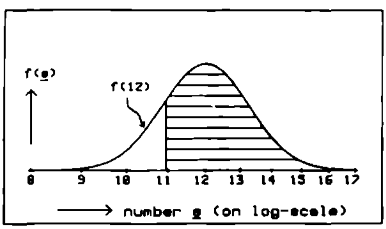

Figure 2. $P_{12,10}(12 \mid 12)$ is the surface under the normal distribution function $(1/\sigma\sqrt{2\pi})\exp - \{[\ln(12) - x]^2\}/2\sigma^2$ extending from ½ln(10 12) to infinity. Again, $\sigma = .1080$.

Defining n_r as that number, larger than n, that can be discriminated from n 50% of the time, we get:

$$Wb = \frac{n_r - n}{n}. \quad (10)$$

The Weber fraction is a constant, so that the relation

$$\frac{n - n_l}{n_l} = \frac{n_r - n}{n} \quad (11)$$

must hold. While n_l is smaller than n, it then easily follows that the limen to the left of n, $n - n_l$, is smaller than that to the right of n $(n_r - n)$. Moreover, a closer examination of the equation stated above leads to the following peculiar relation:

$$n_l \cdot n_r = n^2 \quad (12)$$

or

$$\frac{n_l}{n} \cdot \frac{n_r}{n} = 1. \quad (13)$$

This relation represents well the asymmetric character of number discrimination. For instance, if 25 is the number limen to the left of 30, 36 should be the limen to the right of 30 because $25 \cdot 36 = (30)^2$.

At this point, it is necessary to refer to an early study of Crossman (1956), in which the need for a quantitative measure of "discriminability" was pointed out. He considered two to be discriminated signals S_1 and S_2 as points in a space of one dimension located at the distances x_1, x_2 from the origin. The ease of distinguishing between S_1 and S_2 was expected to depend on the "distance" between x_1 and x_2. In the case of numbers, he did know from experiments that the distance depended on the ratio rather than on the absolute differences between the numbers. So, he took logarithms and measure in the space of log x. The "distance" then became

$$D(S_1,S_2) = |\log x_1 - \log x_2| = |\log \frac{x_1}{x_2}|. \quad (14)$$

This "D function" gives the ease of discrimination between S_1 and S_2,. The reciprocal of D has been named a "confusion function," since it measures the tendency to confuse S_1 and S_2. From our model, it follows that the detectability d′ is $(1/2\sigma)$ log(max/min), which closely resembles the distance function D as formulated by Crossman.

The remainder of this article is a report of two experiments, one threshold experiment (Experiment 1) and one RT experiment (Experiment 2), in which the discrimination hypothesis was tested.

Table 1
Within Each Row Are Schematized Test Numbers t That Are Neighboring Numbers of a Row-Specific s

		t						t				
s−5	s−4	s−3	s−2	s−1	s	s+1	s+2	s+3	s+4	s+5	s+6	s+7
		5	6	7	8	9	10	11				
	8	9	10	11	12	13	14	15	6			
11	12	13	14		16		18	19	20	21		
15	16	17	18		20		22	23	24	25	26	
20	21	22	23		25		27	28	29	30	31	
25	26	27	28		30			33	34	35	36	37

EXPERIMENT 1

Method

Subjects. Four (three males, 1 female) experimentally naive undergraduate psychology students were paid for their participation in the experiment

Stimuli. Dot patterns that differed in numbers of dots, as schematized in Table 1, were constructed. Six numbers (8, 12, 16, 20, 25, and 30) are called standard numbrs, s; the other numbers are called test numbers, t. Within each row of Table 1 is presented one standard s, to the left and right of which are test numbers that are neighboring numbers of a row-specific s. For each number n, 30 different configurations were constructed according to a pseudorandom procedure. This procedure started with partitioning a 10 × 10 square matrix into four quadrants The n dots of a particular stimulus were then placed randomly in the matrix cells with the constraint that each quadrant contained about the same number of dots. Each stimulus was unique with respect to its configuration. The stimuli are illustrated with three examples shown in Figure 3.

Procedure. Participants were tested individually in a quiet laboratory room. The stimuli were presented on a 27 × 20 cm video monitor situated approximately 75 cm from the participant and at eye level. Dot diameter was 2 mm, and the shortest of the distances from one dot to another was 8 mm. Each time a sequence of 60 stimuli, consisting of 30 stimuli with number t and 30 stimuli with a to this t belonging row specific number s (see Table 1) was presented. Following the two-alternative forced-choice method, participants were told in advance what values t and s had, and were instructed to answer "yes" when s was presented and "no" when t was presented. Order of presentation of the 60 stimuli was randomized. Each stimulus was presented for a fixed duration of 100 msec The task was self-paced; a stimulus appeared on the screen 1,000 msec after the participant had pushed a button. The experimenter recorded the participants' responses Whenever a participant committed an error, he or she received immediate verbal feedback ("wrong") from the experimenter. Following the 60 trials, the participant rested a few minutes The next stimulus sequence contained 60 stimuli with a new t and a new s Each participant completed three sessions spread out over 2 or 3 consecutive days, with each session taking about 90 min. The experiment was controlled by a PDP-11/45 system

Results

The percentage of correct responses on any s, presented with each particular alternative t, was calculated over all subjects. The conditional chances, $P_{s,t}$ (s | s) were derived by dividing the percentages by 100. Figure 4 shows these frequencies as a function of (t − s). In Figure 4, one can see that the profiles, each belonging to a particular s, expanded with increasing magnitude of s.

As is usually done, we chose the difference limen at the .75 level, which means that a difference between s and t is noticed 50% of the time. From Figure 4, limens were then measured at the intersection points of the profiles at the .75 chance level. In Figure 5, limens to the left (smaller numbers) and to the right of s (larger numbers) are plotted as a function of s. One can see that for each s the difference limen to the left was always smaller than the limen to the right of s. In Figure 5 the regression lines are also drawn. [Limens for larger numbers (I), rho = .96; limens

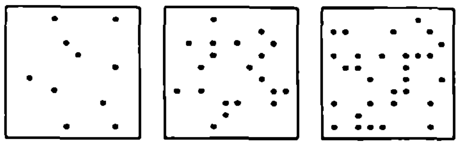

Figure 3. Three examples of stimuli used in the experiments.

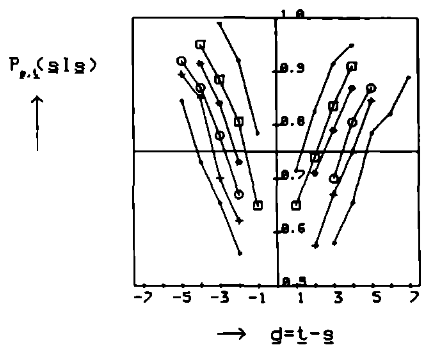

Figure 4. $P_{s,t}$(s | s) as a function of d = t − s, s = 8 ( ), s = 12 (□), s = 16 (•), s = 20 (O), s = 25 (+), s = 30 ( ).

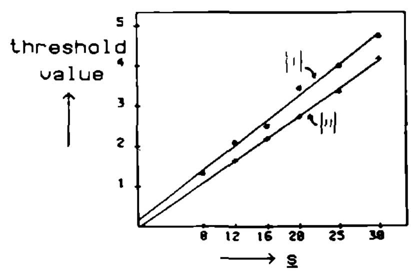

Figure 5. Threshold values to the left (•) and to the right (+) are plotted as a function of s. The corresponding regression lines are also drawn.

for smaller numbers (II), rho = .95]. From regression line II, we derived (max − min)/min = .164. From regression line I, we derived (max − min)/max = .139, from which we calculated that (max − min)/min = .162. We obtained the Weber fraction dS/S = (max − min)/min *as the mean of both values resulting from I* and II: $W_b = .163$. From the relative limens to the right, .164, and to the left, .139, of a particular number, we obtained $(n_l/n)(n_r/n) = 1.164 \times .861 = 1.0022$.

In Figure 6 $P_{s,t}(s \mid s)$ is plotted as a function of max/min. In Equation 8, we postulated a relation between $P_{s,t}(s \mid s)$ and max/min. A likelihood procedure was used here to determine the value of the dispersion σ, given the chances $p_{s,t}(s \mid s)$ and the ratio max/min. A value of .1317 was found for σ. With this value, the chances $P_{s,t}(s \mid s)$ were recalculated as a function of max/min and plotted (solid curve) in Figure 6 as the best fit through the measured $P_{s,t}(s \mid s)$. We then determined the value max/min for which $P_{s,t}(s \mid s) = .75$. This ratio was found to be 1.162, from which the Weber fraction dS/S = (max − min)/min = .162 was easily calculated.

To examine the hypothesis that a normal distribution function should underlie the discrimination process, we plotted the chances $P_{s,t}(s \mid s)$ on a probabilistic scale as a function of $(1/2\sigma)\ln(\max/\min)$ (see Figure 7). This is identical to transforming $P_{s,t}(s \mid s)$ to z scores and then plotting them on a linear scale. It should be noted that $(1/2\sigma)\ln(\max/\min)$ are also z scores. As we can see, there exists a strong linear relation between the transformed $P_{s,t}(s \mid s)$ and $(1/2\sigma)\ln(\max/\min)$, a finding that confirms the normal distribution hypothesis.

The analysis of the results presented above was based upon the assumption that the psychological transform from external stimulus value n to internal representation q_n was logarithmic in nature (i.e., Fechner's law was satisfied). If a less stringent assumption had been made, fulfilling only the property of positive monotonicity, then conjoint measurement methods could be used to specify the nature of the function in question and to test equality of variances. If it is assumed that the cumulative distribution function can be approximated by the logistic function $F(x) = (1 + e^{-x})^{-1}$, then analysis of variance methods can be used to test the additivity properties. Both of these tests would require a factorial design.

The differential sensitivity of visual number can also be studied by letting each stimulus be present on the monitor until the subject gives the response. In this case, any possible strategy such as counting, subitizing, or estimating is applicable. It is reasonable to suppose that the selection of the strategy will be determined largely by whether the difference between two numbers to be discriminated is above threshold. Consider two stimuli comprising 20 and 30 dots. From Experiment 1, we know that they are easy to discriminate from each other. Therefore, one should expect very short latencies as well as very high accuracy. On the other hand, if 29 dots are presented for discrimination from 30, then only the strategy of counting would produce the correct response, a strategy that would increase response time drastically. Thus, somewhere between 20 and 29, there is one number (or more) which, in direct comparison with n = 30, will sometimes be estimated and sometimes be counted. Consequently, one might expect that latencies are distributed according to a bimodal distribution, one peak representing the counting times and the other representing immediate estimation times.

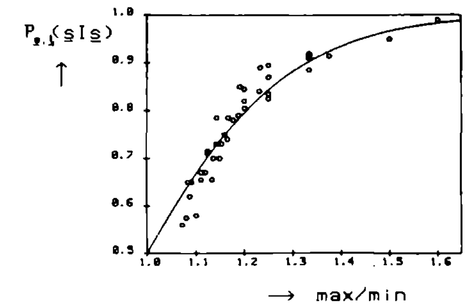

Figure 6. $P_{s,t}(s \mid s)$ plotted as a function of max/min.

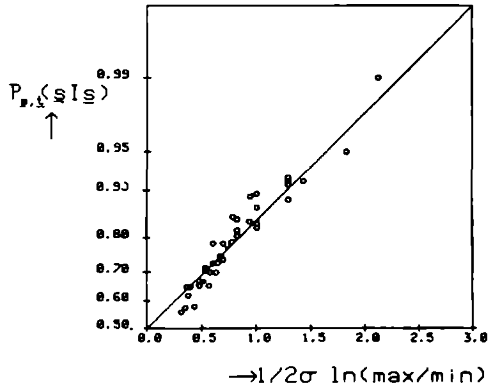

Figure 7. $P_{s,t}(s \mid s)$ plotted on a probability scale as a function of $(1/2\sigma)\ln(\max/\min)$.

EXPERIMENT 2

Method

Subjects. Two young adults took part in the experiment and were paid for their participation.

Stimuli. The stimuli were the same as those use in Experiment 1, with the exception that the standards, s, took the values of 8, 20, and 30. Consequently, only neighboring numbers of these standards served as test stimuli, t (see Table 1).

Procedure. The procedure was in all respects identical to the one followed in Experiment 1, except for the following: Each stimulus now remained visible until the response, mediated by a microphone (Sennheiser headset), had surpassed a previously selected critical level. The subject was instructed to respond with "yes" whenever s was presented and with "no" whenever t was presented. He or she was asked to respond as accurately and as rapidly as possible. Latencies were registered automatically. The experimenter, who had a list of stimulus specifications, scored each response according to whether it was correct or incorrect, and then entered it into the computer. Whenever the participant committed an error, he or she received immediate feedback ("wrong") from the experimenter. Prior to each session, a sequence of stimuli containing only one dot was given. In order to determine the eventual latency differences with respect to verbalization of the response in question, participants responded yes or no in an alternate manner to these single dot stimuli.

Results

The analysis was based on all correct responses given to presented s stimuli. The frequency of incorrect responses was in general not more than 10% and typically occurred to stimuli near threshold. Figure 8 shows, for s values of 8, 20, and 30, the mean reaction times as a function of the difference t − s. As we can see in Figure 8, there was not only a decrease in RT as a function of increase in d, but also marked differences in the absolute magnitude of the latencies. In Figure 9, the standard deviations (SD) of the means depicted in Figure 8 are plotted as a function of d. One can see that, for s = 20 and s = 30, deviations culminated at values of d which were roughly the same as values for the limens derived in Experiment 1. Finally, the distribution of reaction times is illustrated for s as compared with three different test stimuli (t = 23, 25, and 27), which were, respectively, above, around, and below discrimination threshold (Figure 10). While the distribution for the conditions of s,t = (30,23) and s,t = (30,27) point to a unimodal pattern, it is apparently dichotomized in the case of s,t = (30,25).

DISCUSSION

The results of Experiment 2 agree with the findings of Experiment 1, both with respect to the predicted threshold location and the asymmetry of the right- and left-handed thresholds. Threshold values measured in Experiment 2 were, as a rule, slightly smaller than those obtained in Experiment 1, which probably reflects procedural differences between the two experiments. In the situation of Experiment 2, in which the participant had sufficient time to process the visual information and had also been instructed to give the *correct* answer as rapidly as possible, he or she would probably choose a safe strategy (counting) rather than a risky one (estimating). The bimodal shape of the distribution of latencies in those situations in which the difference between the numbers was around threshold suggested that the subject fol-

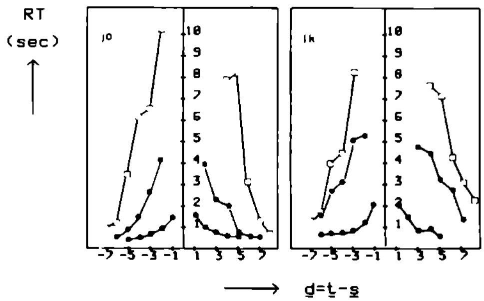

Figure 8. For two subjects, mean reaction time plotted as a function of d = t − s. S = 8 ( ), s = 20 (•), s = 30 (□).

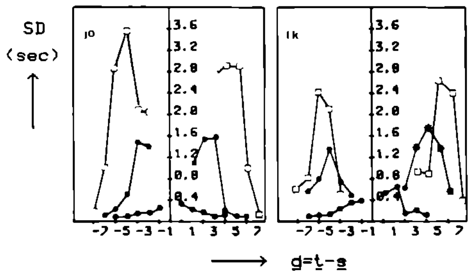

Figure 9. Standard deviations of the means depicted in Figure 8, plotted as a function of d = t − s.

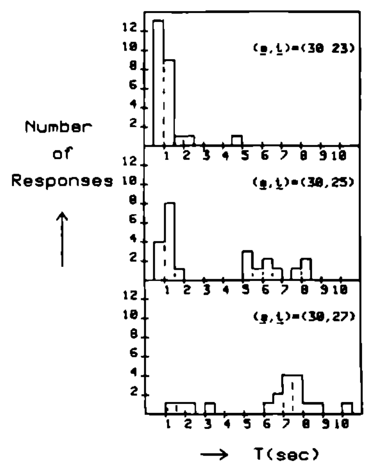

Figure 10. Distribution of reaction times on s = 30 for the conditions s,t = (30,23), (30,25), and (30,27).

lowed a strategy of counting roughly half of the time and a strategy of estimation half of the time. It is also possible, however, that the bimodal shape was affected by shifts in speed-accuracy criteria. In order to control for this rather awkward phenomenon, a much more laborious experimental and analytical design would be needed. Thus, one could vary systematically speed-accuracy instructions for the same task and simultaneously fit frequencies of correct responses and RT data.

The results of Experiments 1 and 2 support Averbach's hypothesis that the span of visual apprehension is limited by the mutual discriminability of visual numbers. Pairwise discrimination between the small numbers 1, 2, 3, 4, 5, and 6 can be done easily, because their ratio (max − min)/min lies well above the Weber fraction of .162. The numbers 6 and 7 can also be discriminated from each other more than 50% of the time, but not 7 and 8. There is confusion between 7 and 8 more than 50% of the time. This result led us to conclude that the number six should be the upper limit of the span of apprehension. However, in an imaginary situation in which only even numbers are possible, the numbers 2, 4, 6, 8, 10, and 12 can all be distinguished from each other over 50% of the time. In that case, the number 12 should be the upper limit of visual apprehension. Moreover, any pairwise discrimination between two numbers of which the ratio (max − min)/min lies above the Weber fraction can be made more than 50% of the time. Therefore, our results strongly support Crossman's (1956) conclusion that the idea that the mind can grasp only a small number of objects at once remains quite unsupported by the evidence, if indeed it has any meaning at all.

So far, we have neglected the influence of pattern on number discrimination. Sometimes a 100% correct identification of four dots has been found (Kaufman et al., 1949), while we found $P_I(4 \mid 4) = .76$, a value deduced from discrimination results of the larger numbers 16 and 20, or 20 and 25. But, for these larger numbers, pattern recognition does not make much sense. Pattern recognition becomes relevant, however, in the case of very small numbers: three dots nearly always make a triangle, for example, and four may often make a recognizable quadrilateral (Neisser, 1966, p. 42). Thus, apart from pure-number discrimination based on probability concepts, discrimination between small numbers can be facilitated by pattern effects.

Larger numbers of dots ($n > 10$) have been shown to be underestimated (Indow & Ida, 1977). In pairwise comparison between two numbers, the discrimination is hardly influenced by underestimation because both numbers are underestimated. However, direct identification should lead to more erroneous results when $n > 10$, so $P_I(n \mid n)$ should then be smaller, as depicted in Figure 1. To avoid these underestimation problems, a correction to our discrimination model can be made by substituting for the objective number n (numerosity) its subjective equivalent m (numerousness). For instance, Indow and Ida showed that, for randomly arranged dot patterns with $n > 10$, the subjective number m was exponentially related to the numerosity n as $m = n^{.87}$.

Both factors, pattern recognition when $n \leq 4$ and underestimation when $n > 10$, do not make much sense when the number of dots is six, so we may safely retain the conclusion that the upper limit of the span of apprehension ($n = 6$) is due to discrimination on a probability basis only.

It is interesting to discuss our discrimination model in relation to the findings of Hunter and Sigler (1940). They showed that it took more light to see two black dots than it did to see one dot. Moreover, to reach a 50% level of correct responses on plates containing two or more dots, successively greater amounts of light were needed. The Bunson-Roscoe law, $I \cdot t = \text{constant}$, was valid here for numbers of dots up to $n = 8$. In our experiments, intensity was far above threshold. The only source of noise we assumed was in the transmission of number information appearing as a constant dispersion, σ, in the internal representation of number. But, if we should decrease intensity, it might well be possible that external noise relatively increases which emerges as an extra noise component, σ_e, in our internal representation, so $\sigma \mapsto \sigma_1 + \sigma_e$. $P_I(n \mid n)$ then decreases monotonically with increasing σ and, therefore, becomes a function of both intensity I and exposure time t. It would be interesting to investigate the proper relationship between the dispersion σ and intensity and/or exposure time.

At this point, we would like to say a word about the end effects that are frequently reported in number-naming tasks. For instance, Averbach (1963) used a limited set of response alternatives, the numbers 1 to 13, and the task was to ascertain the briefly presented numbers of dots. To him, it seemed difficult to explain the constant superiority of 13 over 12 and of 12 over 11, not only in terms of more correct, but in having fewer false alarms as well. In our model, $P_I(n \mid n)$ is the frequency of correct responses to n, while n could be any integer. Limiting the set of alternatives should improve the task for numbers at the end of this limited set. While $P_I(13 \mid 13)$ is represented by the surface of the standard normal distribution function, extending from $(1/2\sigma)\ln(12/13)$ to $(1/2\sigma)\ln(14/13)$, $P_{n=1,\ldots,13}(13 \mid 13)$ is represented by the surface, extending from $(1/2\sigma)\ln(12/13)$ to infinity, so $P_{n=1,\ldots,13}(13 \mid 13) > P_I(13 \mid 13)$. Or, saying it with words, the transformation to internal representation admits a chance of responding 14 when 13 are presented, but the subject knows from instruction (or else, from experience) that there is no 14, so he responds with 13. All possible responses larger than 13 become 13. Of course, this superiority of responding with 13 might bias the response of 12 over 11

Many studies have dealt with the immediate apprehension of number It was assumed that there was some number, n, of discrete objects that the mind could immediately perceive The empirical question, then, was the value of n In general, little discussion was devoted to the actual phenomenon For instance, Beckwith and Restle (1966) merely called the underlying process "a somewhat mysterious but very rapid and accurate perceptual method " Finally, in this study we have shown that "subitizing" (Kaufman et al , 1949), "the span of apprehension" (Averbach, 1963), "the span of attention" (Fernberger, 1921; Freeman, 1912), and "the span of discrimination" (Hunter & Sigler, 1940) all point to the same hypothetical construct, and that they do not refer to different behavioral phenomena Number discrimination, and therefore number identification, is governed by probability concepts operating in a log space. Those numbers for which it mutually holds that $[(\max - \min)/\min] > w_b$, with max and min the larger and smaller numbers in question, can all be discriminated from each other more than 50% of the time. This set of numbers collapses to the classical span of apprehension when the difference between two neighboring numbers is restricted to one These are the small numbers with an upper limit of six, for six can be discriminated from seven above threshold, but seven cannot from eight

REFERENCES

Averbach, E The span of apprehension as a function of exposure duration *Journal of Verbal Learning and Verbal Behavior*, 1963, 2, 60-64

Beckwith, M , & Restle, F Process of enumeration *Psychological Review*, 1966, 73, 437-444

Bourdon, B Sur le temps necessaire pour nominer les nombres *Revue Philosophique de la France et de l'Etranger*, 1908, 33, 426-431

Crossman, E R F W The measurement of discriminability *Quarterly Journal of Experimental Psychology*, 1956, 7, 176-195

Fernberger, S W A preliminary study of the range of visual apprehension *American Journal of Psychology*, 1921, 32, 121 133

Freeman, F N Grouped objects as concrete basis for the number idea *Elementary School Teacher*, 1912, 12, 306 314

Hunter, W S , Sigler, M The span of visual discrimination as a function of time and intensity of stimulation *Journal of Experimental Psychology*, 1940, 26, 160-179

Indow, T , Ida, M Scaling of dot numerosity *Perception & Psychophysics*, 1977, 22, 265 276

Kaufman, E L , Lord, M W , Reese, T W , & Volkman, J The discrimination of visual number *American Journal of Psychology*, 1949, 62, 498-525

Moyer, R S , & Landauer, T K The time required for judgments of numerical inequality *Nature*, 1967, 215, 1519-1520

Neisser, U *Cognitive psychology* New York Appleton Century-Crofts, 1966

Parducci, A Range-frequency compromise in judgment *Psychological Monographs*, 1963, 77 (2, Whole No 565)

Taves E H Two mechanisms for the perception of visual numerousness *Archives of Psychology*, 1941, No 265

Thurstone L L A law of comparative judgment *Psychological Review*, 1927, 34, 273 286

Thurstone, L L Fechner's law and the method of equal-appearing intervals *Journal of Experimental Psychology*, 1929, 12, 214-224

Torgerson W S *Theory and methods of scaling* New York Wiley, 1958

Woodworth, R S , & Schlosberg, H *Experimental psychology* New York Holt, Rinehart & Winston, 1954

(Manuscript received November 16, 1981, revision accepted for publication April 14, 1982)

Patt. Recog. Vol. 16, No. 3, pp. 341–348, 1983

AN ALGORITHM FOR PATTERN DESCRIPTION ON THE LEVEL OF RELATIVE PROXIMITY

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(*Received 25 March 1982, in revised form 13 October 1982, received for publication 7 January 1983*)

Abstract This paper presents a purely data-driven and knowledge free cluster algorithm which formalizes the Gestalt rule of proximity as it is encountered in dot figures. The concept of neighbourhood with respect to relative proximity is described by a binormal distribution function. This spread function expresses the strength with which a point operates upon its surroundings. Summing all partial functions yields a primary description of the dot figure which can be viewed as a landscape with mountains where dots are relatively close to one another and valleys where they are relatively far apart from each other. Clusters boundaries and shapes emerge in regions where the resultant function surpasses a threshold value. The proposed method is discussed on the basis of a few examples. The method is further discussed in relation to smoothing techniques for the visual enhancement of noisy patterns and to propagation models for the description of visual form.

Cluster-algorithm Gestalt Relative proximity Parzen estimators Binormal distribution Primal sketch Threshold Clusters boundaries and shapes

Gestalt theory is notably concerned with properties of perceived figures. For instance, while visual figures may differ considerably regarding the nature of their constituent parts, the global organization of the parts (in each of them) can still result in the same perceptual interpretation (see Fig. 1). Such figures are conceptualized as equivalent patterns.

It is known that the patterning of a figure is governed by such Gestalt rules as those of proximity, good continuation and similarity. These rules have been generally accepted as important guidelines for the understanding of how figures are perceptually organized.[1] However it has been repeatedly noticed that their predictive power remains weak as long as one cannot formulate them in operational terms which allow for quantitative predictions instead of informal *ad hoc* demonstrations. Even obvious formalizations of proximity in such simple figures as dot configurations, described by various cluster algorithms, are essentially heuristic and demand suitably chosen criteria to segregate groups of dots.[2,3,4,5] This paper is concerned with a cluster technique which should overcome, at least partly, several of the problems of an intuitive or heuristic approach.

For clarity's sake it is useful first to distinguish between perceptual-level and cognitive-level techniques.[6] Computer realization of perceptual processes encompass the extraction of features and the detection of simple objects. Based on proximity considerations, neighbourhood concepts are used to describe clusters, boundaries and shapes. On the other hand, cognitive-level techniques deal mostly with formal aspects of picture-syntax and scene-analysis insofar as they are based upon symbol structure manipulations. These latter methods depend heavily on the availability of knowledge, based upon past experiences stored in (long-term) memory. However, it is hard to establish a human's past experience and, therefore, it seems difficult to attack the problem of pattern recognition by searching for cognitive solutions.

Perceptual psychologists[7,8] have emphasized the idea that there exist early stages in visual perception that are not determined by the perceiver's knowledge of the semantic aspects of the visual input. Marr[7] argued that a very great deal of information may in fact be extracted from images with knowledge-free techniques. Grouping considerations based on orientation,

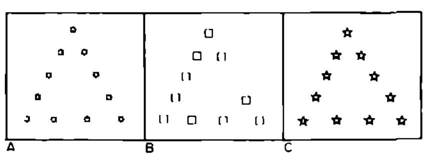

Fig. 1. Examples of figures with equal global organization but different constituent parts.

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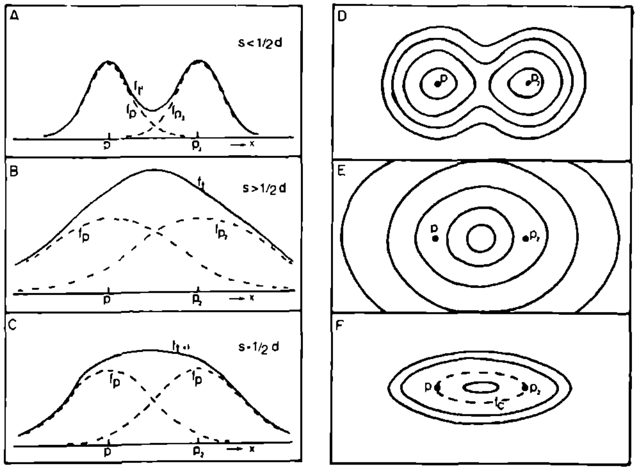


Fig 2 $f_p(x)$, $f_{p_1}(x)$ and $f_t(x)$ are plotted as a function of x, for $s \ll 1/2d$, $s \gg 1/2d$ and $s = 1/2d$, respectively (Figs 2a, 2b and 2c). In Figs 2d, 2e and 2f are depicted the two dimensional representations in terms of iso-lines

proximity, gray-level, color, etc. would provide a computation of a primal description of the image (the so-called "primal sketch"). This primal sketch results from a purely data-driven and context-free algorithm. Marr further suggested that the image-formation may be described mathematically in terms of degradation operators or spread functions which characterize the optical channel.

The method (CODE) proposed in this study involves a procedure in which a binormal distribution function is superposed on each element of a dot figure. The function should represent the strength with which an element operates upon its surroundings. The concept of neighbourhood in terms of relative proximity is expressed by the width of the spread function. This dispersion is fixed by the distance between each element and its nearest neighbour. Clusters are then formed in regions where the sum of the partial functions reaches a threshold value. Before discussing the method in more detail, a similarity with the procedure of the Parzen estimation method for probability density functions will be indicated.

The problem of estimating a probability density function $\hat{F}(x)$ from a number of randomly selected samples $x_1, x_2, \ldots, x_n$ has played an important role in the field of pattern recognition. If little *a priori* knowledge about $\hat{F}(x)$ is available, a non-parametric estimate of $\hat{F}(x)$ could be useful. A well-known method is the Parzen[9] estimation,

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{i=n} h^{-m}(n)\, K\left(\frac{\|x - x_i\|}{h(n)}\right) \qquad (1)$$

in which K, the kernel is an arbitrary bounded probability density, n is a number of samples, $h(n)$ are positive numbers, m denotes dimensionality, and $\|\ \|$ is the Euclidean norm. Examples of K are $\exp(-\|x\|)$ and $\sin^2(\|x\|)/\|x\|^2$. Much attention has been given to the choice of the "smoothing factor" (standard deviation in the case of normal density)[10,11]. As can be understood from (1), the degree of smoothing of the estimate $\hat{F}(x)$ is controlled by $h(n)$. For that reason, h is called the smoothing parameter. A standard deviation related to the nearest neighbour has been described[12,13]. The just mentioned methods have been proposed mainly for automatic classification in the fields of pattern recognition and statistical analysis. It is the purpose of this study to extend these techniques to visual scene analysis and to use them as a tool for the quantification of Gestalt principles of human perceptual organization.

To explain the method CODE in more detail, the simplest case of clustering of a set of only two dots will now be treated. We can represent the dots (labeled p_1 and p_2) in one dimension with coordinates x_1 and x_2, respectively. For the simultaneous presentation in the perceptual field of only two dots we assume the clustering strength to be optimal. For the case of simplicity any restriction to the boundaries of the stimulus field is left out of consideration. In order to describe the clustering of p_1 and p_2, a normal distribution function

$$f_{p_i}(x) = \exp -\frac{(x_i - x)^2}{2s_i^2}$$

(further defined as $[x_i, s_i]$)

is superimposed on each of the two dots $f_{p_1}(x) = [x_1, s_1]$ and $f_{p_2}(x) = [x_2, s_2]$, as illustrated in Fig 2. The strength with which a dot influences its surround-

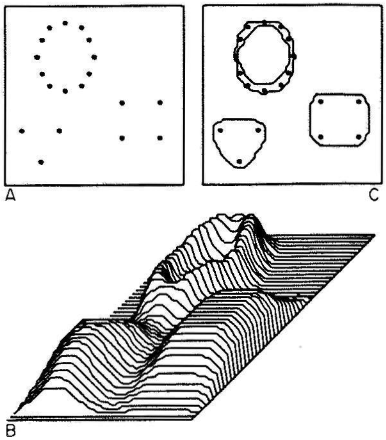


Fig. 3. Figure 3b shows, as an illustration of the concept of relative proximity, the primal description of the dot figure depicted in Fig. 3a. The perceptually relevant cluster boundaries (contours) are depicted in Fig. 3c.

ings is reflected by the property of the normal function: the effect is maximal at its own position and monotonically decreases to zero towards its peripheral surroundings. The range of a point's influence is fixed by the standard deviation or dispersion, s. This parameter must be selected to yield an optimal representation of clustering strength. In relation to p_1 and p_2, it is clear that s_1 equals s_2 from the point of view of symmetry, so that $s_1 = s_2 \equiv s$. When the dispersion is given a very small value with respect to the distance d between p_1 and p_2, the summed function $f_t(x) = [x_1, s] + [x_2, s]$ will show a minimum just between p_1 and p_2 (Fig. 2a). In that case $f_t(x)$ under-represents the assumed strength of clustering. However, when the dispersion is large in comparison with d the resultant function $f_t(x)$ will show a sharp peak between p_1 and p_2. In this case, $f_t(x)$ over-represents the clustering of p_1 and p_2 (Fig. 2b). It was decided therefore to take that value for s for which $f_t(x)$ just reaches a maximum, which emerges when $s = 1/2d$ (Fig. 2c).

The three clustering representations depicted in Figs 2a, 2b and 2c are in one-dimensional form. Expanding them to a two-dimensional form yields a description of the field of influence in terms of contours (see Figs 2d, 2e and 2f), which are isolines of the two-dimensional function $f_t(\bar{x})$. In order to distinguish between the one-dimensional and two-dimensional situation we now switch from scalar to vectorial notation. In this way, a contour is defined as that set of points ($\mathbf{x}$) for which it holds that $f_t(\mathbf{x}) = \text{constant}$.

We now proceed to the case of a set of n dots, labeled $p_1, p_2, \ldots, p_n$ with positions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively. A second assumption is made in these cases. That is, with the function $f_{p_i}(\mathbf{x})$ of a particular dot p_i in the set, the dispersion equals half the distance of that element to its nearest neighbour $s_i = 1/2d(\mathbf{x}_i, \mathbf{x}_j)$, with $\mathbf{x}_j$ equalling the position of p_j, the nearest neighbour of p_i. In this way the strength of a point's influence upon its surroundings depends on the distance to its nearest neighbouring point. Once the dispersion value, and therefore the distribution function, has been established for each dot of the set, all partial distribution functions are summed

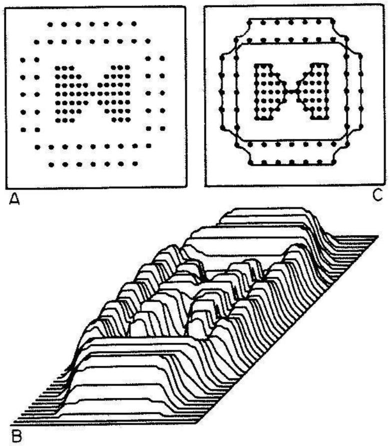
$$(f_t(\mathbf{x}) = \sum_{i=1}^{i=n} f_{p_i}(\mathbf{x})).$$

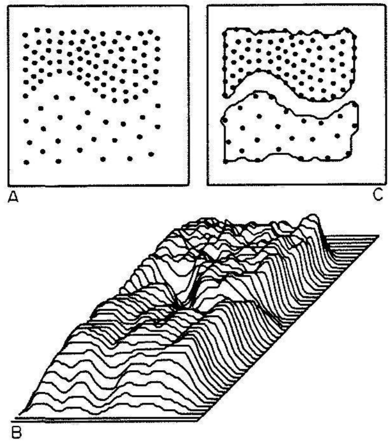
The resulting landscape of mountains and valleys can be seen to be quite akin to Marr's[7] primary sketch. Figure 3b presents an illustration of such a landscape. The function $f_t(\mathbf{x})$, which is the result of the method applied to the dot figure depicted in Fig. 3a, is plotted as a function of $\mathbf{x}$.

Given a primal sketch, we now come to the question of how to extract clusters. If a dot is situated such that its distance to other dots is very large while these other dots have much smaller mutual distances, then the dot should be recognized as an isolated one. CODE adds to this dot a binormal distribution function with a high dispersion. Contributions of binormal distributions of other dots are (almost) negligible near the isolated dot. A threshold value f_0, being the peak value of the binormal spread function, will now produce an (almost) infinitely small cluster area, indicating the existence of an isolated dot. A contour corresponding to this threshold value is now defined as that set of points ($\mathbf{x}$) for which $f_t(\mathbf{x}) = f_0$, with f_0 being the peak value of the binormal spread function. A cluster is then defined as the subset of all dots which lay precisely on or within that particular contour. For the dot figure depicted in Fig. 3a the relevant contour specified by the relation $f_t(\mathbf{x}) = f_0$ splits up the dot figure into one quadrilaterally shaped cluster, one that is triangularly shaped and one cluster which is shaped as a ring (see Fig. 3c).

It is important to notice that clusters are formed in regions where dots are mutually near neighbours rather than simply near neighbours. Since CODE considers only relative proximity between dots rather than absolute distances, it is invariant under similarity transformations (translation, rotation and changes of size). Also, CODE is insensitive to the order in which the points of a figure are scrutinized. For computer purposes CODE has been inserted into a program which treats the coordinates of the dots of a figure as input. The stimulus field was divided into a 50 × 50 matrix. For each cell the contribution of all partial distribution functions was calculated. In converting from real to integer values the programming might deviate a bit from the one that is described earlier. In the following, CODE is illustrated with a few examples. It will give us the opportunity to discuss the validity of the proposed method.

The concept of relative proximity is exemplified by Fig. 3. For instance, the nearest neighbours of the triangular point to the right could be the two other





Figs 4 and 5. Two examples to illustrate the cluster technique applied to the problem of detecting a sharp gradient in point density. The dot figure presented in Fig. 5a is from Zahn.[5]

triangular points as well as the most adjacent circular point. This last point, however, finds its nearest neighbours in points appearing on the circle. As a consequence, the mutual nearest neighbour property of CODE results in a strong inter-relationship for the triangular points apart from a strong relation from the circular points. Note that for the circularly arranged points the predicted surface emerges as a ring over the points, while for the triangular points (as for the quadrilateral) the surface emerges as an area enclosed by an equilateral triangle (or square).

Figures 4 and 5 are presented to illustrate the cluster technique applied to the problem of detecting a sharp gradient in point density between two fairly homogeneous areas of different density. In particular, Zahn[5] noted that in Fig. 5a one readily perceives a boundary along the area of highest density. Zahn's observation may be amplified by noticing that two boundaries are readily perceived, namely one along the area of highest density and one along the area with a smaller density of dots. The perception of an empty space between these boundaries is obvious. Figure 5c confirms this observation very well.

The point set depicted in Fig. 6a is a cluster problem referred to by Arkadev and Braverman[14] and later by Zahn.[5] Zahn treated this problem using graph theoretical methods. The notion that humans perceive Fig. 6a as two clusters joined by a small neck urged Zahn to enrich graph theoretical techniques with a new definition of a "neck in a graph". As can be seen in Fig. 6c, CODE has no difficulty discovering a contour which is the outline of two clusters (blobs) joined by a small neck.

CODE has proved to be a reliable tool for the description of pattern-number interaction in number naming tasks on visually presented dot stimuli.[15] Traditional experimental research on this subject included tachistoscopic experiments on visual numerosity judgments[16,17,18] as well as reaction time experiments in which the time was measured to ascertain a presented number of dots.[17,19,20] From the functional relation between either median response-time or frequency of correct response and absolute number, two number processing strategies were proposed: one for the immediate apprehension of small numbers of dots up to a maximum of about six dots, a process which later was called "subitizing"

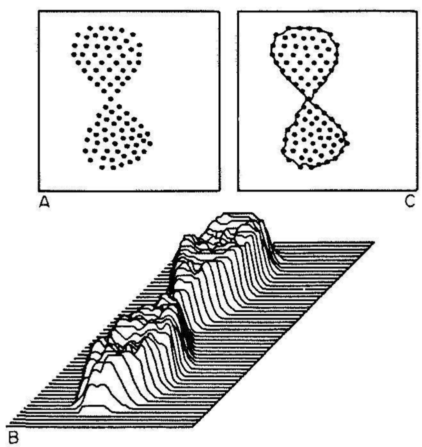


Fig. 6. CODE applied to a cluster problem referred to by Arkadev and Braverman[14] and later by Zahn.[5]

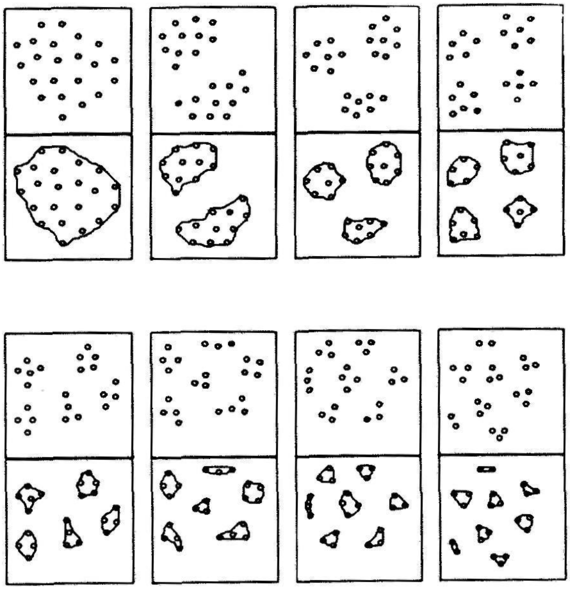
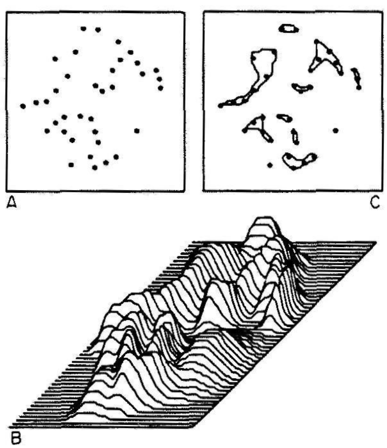
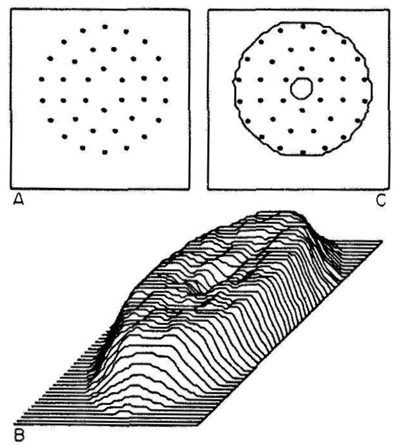


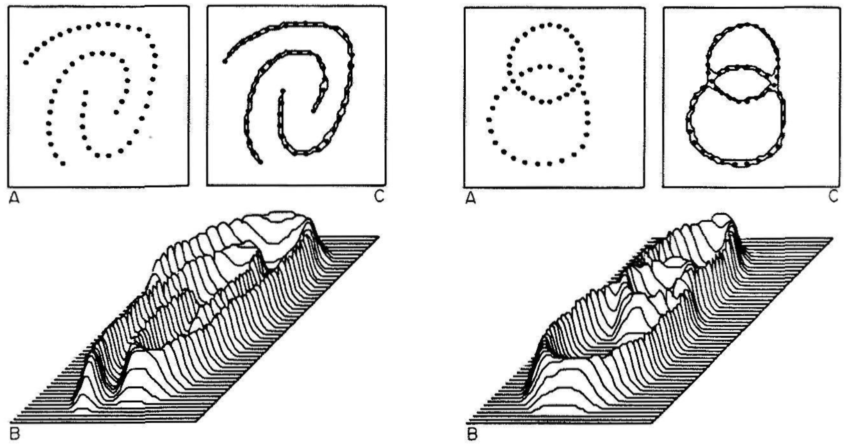
Fig. 7. Eight dot figures, with the same number of dots (n = 22) but with a different number of groups (g = 1–8), used in experiments by van Oeffelen and Vos.[15]

(after Kaufman *et al.*[17]), and one process operating on larger numbers of dots, simply called "counting". In their experiments the previous mentioned authors purportedly cancelled out the factor of patterning by the use of several differently scrambled dot patterns for the presentation of one and the same number. However, even so-called random dot patterns do have a structure, although not a simple one, and these patterns may be perceived as being segmented into groups.[21] It seems dubious, therefore, to base conclusions solely on measures of central tendency in response-time, without discussing or even specifying dispersion values. In a reaction-time experiment, van Oeffelen and Vos[15] used dot figures which differed





Figs 8 and 9. CODE applied to a randomly arranged dot figure (Fig. 8a) and to a regularly arranged one (Fig. 9a). The dot figures are similar to the kind used by Ginsburg.[23]



Figs 10 and 11. An illustration of the validity of the threshold value $f_t(x) = f_0$. A restraint of CODE is illustrated by Fig. 11c.

both in number of dots (n = 14–23) and arrangement of dots. The factor of patterning at the level of grouping-by-proximity was described by CODE. Figure 7 depicts eight of the dot figures used in the experiment, with the same number of dots (n = 22) but with a different number of groups.

The results of the experiment largely confirmed CODE-based predictions and thereby indicated that large collections of dots are preferably counted by groups. Small ($n \leq 5$) groups were subitized and partial results summed to a running total. Based on criteria other than dot proximity, large ($n > 5$) proximity-based groups were subdivided into smaller groups of two or three dots, which again are subitized.

The aspect of pattern–number interaction in visually presented dot figures has also been mentioned in studies on judgments of numerousness. Taves[22] reported that random patterns of dots appear more numerous than the same number of dots arranged regularly on the rim of a circle. Later on, Ginsburg,[23] using a method of simultaneous comparison, found that regular patterns of 37 dots appeared more numerous than random patterns of the same number. Ginsburg related his finding to an explanation proposed by Birnbaum and Veit[24] who suggested that judgments of variables such as numerousness depend in part on a contrast between the numerosity perceived and that expected from other cues. If one would expect random patterns to be more numerous, then the same number would be judged less numerous. In Figs 8 and 9 are depicted two dot figures similar to the kind used by Ginsburg.

Both the randomly arranged dots and the regularly arranged ones were subjected to CODE, resulting in two different patterns; the random dot figure falls apart into small clusters with large empty spaces in between (Fig. 8c), while the regular figure results in one large cluster (Fig. 9c). In addition to the expectancy–contrast explanation of Birnbaum and Veit it could well be possible that differences in perceived subjective surface can account for under-estimating the randomly arranged dots with respect to the regularly arranged ones. A similar explanation can be given to experimental results reported by Krueger.[25] In his experiments, dots appeared less numerous when bunched together on a sheet than when spread out over a larger area.

Figures 10 and 11 are presented to show that the threshold value according to which contours emerge ($f_t(x) = f_0$) is chosen correctly. For instance, the dot figure depicted in Fig. 10a emerges as two twisting curves (see Fig. 10c), the surfaces of which are neglectable. The dot figure depicted in Fig. 11a demands for a same characteristic, though Fig. 11c also illustrates the restriction of the clustering method. CODE cannot extract two circles from the background though perception of two overlapping circles is clear. We will discuss this restraint further on.

CODE resembles in some way smoothing, a technique which is widely used in visual enhancement of noisy dot patterns. Smoothing involves luminance averaging of adjacent areas, and, as such, worsens resolution. Enhancement of global form aspects of a figure is aimed at, at the expense of local detail. In particular, the method of variable spatial averaging[26] allows one to vary the degree of smoothing dependent on local needs for it and to degrade the resolution as little as possible. In regions of low luminance, averaging over a large area will degrade resolution only slightly. On the other hand, in regions of high luminance only a small amount of smoothing is desirable. Connected shapes then appear after some local smoothing has occurred In practice smoothing is used as a direct manipulation of the grey-scale in dot figures consisting of hundreds or thousands of dots rather than small numbers of dots

Another related method was proposed by Blum[27] In his attempt to find possible physiological mechanisms for the extraction of global shape information by the animal visual system, he hypothesized a propagation or diffusion model Blum was inspired by inspection of the primitive visual system, where at the retinal level firing activities of stimulated cells cause neighbouring cells to fire slightly later Blum suggested an analogy between his propagation model and the spreading of a fire in a field of grass The set of points (retinal cells) where the firings gather from different directions is called the "medial axis" of the pattern The medial axis function", then, is the functional relation between each point of the skeleton and its distance from the pattern This function would allow a quantified comparison of a large set of geometrical forms A similar field type of theory has been proposed by Bitterman *et al*[28] They tried to explain form thresholds by invoking processes analogous to the diffusion or growth patterns of bacterial colonies

Navon[29] put forward the idea that perceptual processes are temporarily organized so that they proceed from global structuring towards more and more fine-grained analysis In other words, a scene is decomposed rather than built up It follows that global features of a visual object will be apprehended before its local features CODE treats the spatial organization of a dotted figure as a sort of crude figural analysis which leads to a primal sketch as a first globalistic impression Cluster detection clearly is object extraction Therefore, CODE deals with the problem of finding boundaries between object and background in dot figures It seems that CODE can adequately detect clusters in dot figures through the recovery of their perceptual boundaries The articulation of their shapes is determined purely by accentuating the idea of relative proximity in relation to the optimal clustering in the two-point figure This criterion sharpens the figure ground relationship insofar as contour bounded clusters are removed from the background However, groupings can be perceived as background material, as one observes in Fig 11c, though which of the two circles is figure and which ground seems ambiguous We hope that a directional constraint, which should be contextual, can be implemented in CODE For instance, a preferred direction for stretching out a points influence can easily be inserted through desymmetrizing the dispersion value s in the spread function f This change would lead to a formal description of both proximity and continuity

SUMMARY

A purely data-driven and knowledge-free clustering method has been developed which formalizes the Gestalt rule of proximity as it is encountered in dot figures The method (CODE) involves a procedure in which a binormal distribution function is superposed on each element of a dot figure The function should represent the strength with which an element operates upon its surroundings The concept of neighbourhood in terms of relative proximity is expressed by the width of the spread function This dispersion is fixed by the distance between each element and its nearest neighbour Clusters are then formed in regions where the sum of the partial functions reaches a threshold value

It is indicated that CODE closely resembles the procedure of the Parzen estimation method for probability density functions proposed for automatic classification in the field of pattern recognition and statistical analysis The psychological validity of the method is discussed on the basis of a few examples The method is further discussed in relation to smoothing techniques for the visual enhancement of noisy patterns and to propagation models for the description of visual form Finally, it is suggested how CODE can be elaborated so as to handle the Gestalt rules of both proximity and good continuation

Acknowledgements This research was supported by a grant to Dr P G Vos from the Netherlands Organization for the Advancement of Pure Research (Zuiver Wetenschappelijk Onderzoek) The authors wish to thank an anonymous reviewer Portions of this paper benefitted from discussions with Dr Michael Katzko

REFERENCES

1 I Zusne *Visual Perception of Form* Academic Press, New York (1970)
2 J F O Callaghan Human perception of homogeneous dot patterns *Perception* **3**, 33 45 (1974)
3 K Chidananda Gowda and G Krishna Agglomerative clustering using the concept of mutual nearest neighbourhood *Pattern Recognition* **10**, 105 112 (1978)
4 W E Wright Gravitational clustering, *Pattern Recognition* **9**, 155-166 (1977)
5 C T Zahn Graph theoretical methods for detecting and describing Gestalt clusters, *IEEE Trans Comput* **C-20**, 68-86 (1971)
6 P V Sankar, C U Sharma and R Narasimhan, Computing the organization and shapes of two-dimensional dot patterns A perceptual level approach, *Comput Graphics Image Process* **8**, 203-218 (1978)
7 D Marr, Analyzing natural images, a computational theory of vision MIT AI Lab AI Memo 334 (1975)
8 W Zucker, A Rosenfeld and L S Davis, General purpose models expectations about the unexpected Technical Report, University of Maryland Computer Center (1975)
9 E Parzen On the estimation of a probability density function and the mode, *Ann math Statist* **33**, 1065-1976 (1962)
10 D O Lofsgaarden and C D Quesenberry, A nonparametric estimate of a multivariate density function, *Ann math Statist* **36**, 1049-1051 (1965)
11 R P Duin, On the choice of smoothing parameters for Parzen estimators of probability density functions, *IEEE Trans Comput* **C-25**, 1175-1179 (1976)
12 L Breiman, W Meisel and E Purcell, Variable kernel estimation of multivariate densities, *Technometrics* **19**, 135-144 (1977)

13 J W Raatgever and R P Duin, On the variable kernel method for multivariate nonparametric density estimation *Compstat 1978 Proceedings in Computational Statistics* L Corsten and J Hermans, eds, pp 524 533 Physica Verlag, Wien (1978)

14 A G Arkadev and E M Braverman, *Computers and Pattern Recognition* Thompson Book Co, Washington D C (1967)

15 M P van Oeffelen and P G Vos, Configurational effects on the enumeration of dots counting by groups, *Memory Cognit* **10**, 396–404 (1982)

16 W S Jevons, The power of numerical discrimination, *Nature* **3**, 281 282 (1871)

17 E L Kaufman, H W Lord, T W Reese and J Volkman, The discrimination of visual number, *Am J Psychol* **62**, 498 525 (1949)

18 I J Saltzman and W R Garner, Reaction time as a measure of span of attention, *J Psychol* **25**, 227 241 (1950)

19 E M Jensen, E D Reese and T W Reese, The subitizing and counting of visually presented fields of dots, *J Psychol* **30**, 363–392 (1950)

20 D Klahr, A production system for counting, subitizing, and adding, *Visual Information Processing* W G Chase, Academic Press, New York (1973)

21 W R Uttal, *An Autocorrelation Theory of Form Detection* John Wiley and Sons, New York (1975)

22 E H Taves, Two mechanisms for the perception of visual numerousness, *Archs Psychol N Y* No 265 (1941)

23 N Ginsburg, Perceived numerosity, item arrangement, and expectancy, *Am J Psychol* **91**, 267–273 (1978)

24 M H Birnbaum and C T Veit, Judgmental illusion produced by contrast with expectancy, *Perceptn Psychophys* **13**, 149 152 (1973)

25 L E Krueger, Perceived numerosity *Perceptn Psychophys* **11**, 5–9 (1972)

26 S M Pizer and H G Vetter, Perception and processing of medical radioisotope scans, *Pictorial Pattern Recognition* pp 147–156 Thompson Book Co, Washington D C (1968)

27 H Blum, A transformation for extracting new descriptors of shape *Models for the perception of speech and visual form* W Wathen-Dunn, ed pp 362 380 MIT Press, Cambridge, Massachussetts (1967)

28 M E Bitterman, J Krauskopf and J F Hochberg, Threshold for visual form a diffusion model, *Am J Psychol* **67**, 205 219 (1954)

29 D Navon, Forests before trees the precedence of global features in visual perception, *Cognitive Psychol* **9**, 353 383 (1977)

Memory & Cognition
1982, Vol 10(4), 396-404

Configurational effects on the enumeration of dots: Counting by groups

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The processing time for quantifying numerosity of two-dimensional dot patterns was investigated as a function of both number of dots and relative proximity between dots. A cluster algorithm (CODE) was first developed as a formal model of how human subjects organize neighboring dots into groups. CODE-based predictions of grouping effects on number processing latencies were then tested with patterns consisting of n dots (range n = 13-23). The results largely confirmed CODE-based predictions and thereby indicated that large collections of dots are preferably counted by groups. Small (n < 5) groups are subitized and their partial results are summed to a running total Based on criteria other than dot proximity, large (n > 5), proximity-based groups are subdivided into smaller groups of two or three dots, which are again subitized

It is intuitively clear that the time required to ascertain the exact number of objects in a given set depends not only on the magnitude or size of the set, but also on the arrangement of objects (i e , their patterning) This holds in particular for numbers beyond the span of immediate apprehension, also called subitizing (after Kaufman, Lord, Reese, & Volkman, 1949) Thus, a set of 20 objects is likely to take much more processing time when it is patterned according to a uniformly spaced row than when it is a two-dimensional pattern segmented into five groups of 4 objects each The row can hardly be enumerated by a procedure other than counting one by one The other pattern can be processed by subitizing the five groups and summing those results in four steps of simple addition Moreover, counting of the row is easily hampered by lateral interference of neighboring dots during the processing of a particular dot, whereas such difficulty is not so apparent with the second pattern

Although the interactive effect of number and patterning on the processing of visual number has been explicitly noted by earlier authors (see Woodworth & Schlosberg, 1954, chap 4), remarkably few later studies have paid attention to it Rather, the patterning factor was purportedly canceled out by the use of several differently scrambled dot patterns for the representation of one and the same number However, even so-called random dot patterns may be perceived as being segmented into groups Such a perception may be caused by, among other factors, relative differences in interdot proximity In some of these stimuli, moreover, the size of the groups will be within the subitizing span, whereas others will exceed this range It seems dubious, therefore, to base conclusions about the latency-numerosity function solely on measures of central tendency, without discussing or even specifying dispersion values (e g , Beckwith & Restle, 1966, Klahr, 1973, Klahr & Wallace, 1976)

At least one latency experiment (Klahr, 1973) supports our assertion that grouping of even random dot patterns must be more adequately controlled for than it has been in past research Briefly summarized, Klahr wanted to know whether number-naming latencies were affected by pattern density or visual angle of a display of dots in the range of n = 1-20 He presented stimuli under two visual conditions, one in which the dots were randomly clustered in the center of the display ("inner condition") and the other in which they were randomly spread out in the periphery of the display ("outer condition") (Figure 1) Differences between mean latencies for outer and inner conditions were calculated, and it was concluded that no consistent relationship emerged with respect to the values of n ⩾ 7 A closer inspection of Klahr's data, however, showed that, with two exceptions, the outer condition stimuli were processed faster than the stimuli of the inner condition This can be explained by the higher probability of the "outer" stimuli to be perceived as a set of proximity-based, small, and subitizable clusters. The "inner" stimuli much less favor such segmentation, in other words, they are predominantly seen as a cluster the size of which remains beyond the subitizing range Klahr's results indicate that stimulus groupability affects number-naming latencies This effect raises the questions of whether and how one can predict the course of latencies for numbers larger than, say, n = 6 as a function of the interaction between set size and patterning

Bourdon (1908) was probably the first experimenter to manipulate the factor of patterning in one-dimensionally arranged dot figures He concluded from

This research was supported by a grant from the Netherlands Organization for the Advancement of Pure Research (Zuiver Wetenschappelijk Onderzoek)

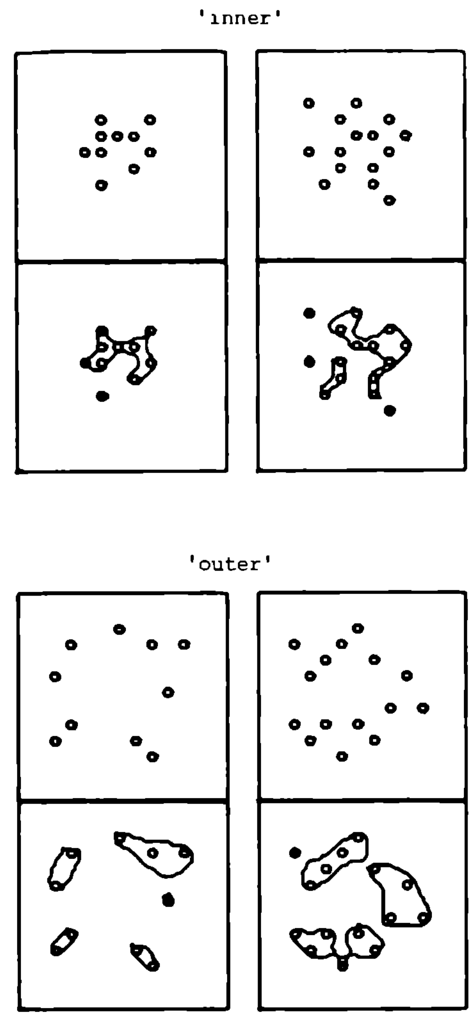

Figure 1 A few typical patterns in the "inner" and "outer" conditions, used by Klahr (1973), are presented CODE predicted clusters are shown below these patterns

experimental results that the perception of number presupposes at least two psychological operations (1) the perception of units of which the total number of dots is composed and (2) the grouping of these units Much later, Atkinson, Cambell, and Francis (1976a, 1976b) varied the interdot interval along an array of dots so that a number of dots were clearly seen as a single entity whereas other dots were seen in another group Atkinson et al related the "chunking" of information to the bandwidth of channels tuned to a particular size or spatial frequency In an experiment, they showed that if two parts of a dotted figure were discriminable by an intergroup space of three times the interdot interval then accurate counting within each subgroup up to a maximum of four could take place

Also, Beckwith and Restle (1966) manipulated the factor of patterning, and they successfully showed that the interaction in question does exist In the discussion of their experiments they referred to Gestalt theoretical rules of proximity, good continuation, and similarity as a basis for understanding pattern number interaction Although these rules have been generally accepted as important guidelines for understanding how visual forms of patterns are perceptually organized (Zusne 1970), their predictive power remains weak as long as one can not formulate them in operational terms that allow for quantitative predictions instead of informal ad hoc demonstrations

The present study is a first attempt to develop a predictive model about the interaction between patterning and number of dots during the processing of numbers in the range beyond the subitizing span We first propose an algorithm, called CODE that is a formal description of how proximity dependent grouping takes place in stimuli like those exemplified in Figure 1 After an experimental test of the perceptual validity of CODE, we elaborate a number of equations for number response latencies as a function of both set size and within-set groupability The subsequently derived hypotheses are then tested in a latency experiment

CODE A CONTOUR-DETECTING CLUSTER ALGORITHM

According to Gestalt theory, the perceptual interpretation of a figure is governed by, among other principles, the principle of proximity A person tends to group those elements of a figure that are close to each other Attempts to formalize grouping according to proximity have been made since the early 1960s, when computer facilities became progressively available However, these various cluster algorithms are less appropriate for studies in visual pattern perception First, many of these algorithms have arbitrary stop rules That is, the level of clustering that optimally reflects perceptual grouping is determined largely by intuition Second, most of these algorithms were developed to handle very large sets of elements and are not well suited to finding clusters in small sets

Perceptual psychologists (e g , Marr, Note 1, Zucker, Rosenfeld, & Davis, Note 2) have emphasized the idea that there exist early stages in visual perception that are not determined by the perceiver's knowledge of the semantic aspects of the visual input Marr (Note 1) argues that a great deal of information may in fact be extracted from an image by means of knowledge free techniques. Grouping considerations based on orientation, proximity, gray level, color, and so on, provide a computation of a primal description of the image (the so-called "primal sketch"). This primal sketch should be a result of a purely data-driven and context-free algorithm. The image formation may be described mathematically in terms of degradation operators or spread functions that characterize the optical channel.

The method proposed in this study involves a procedure in which a binormal distribution function is superposed on each element of a dot figure. This function should represent the strength with which an element operates upon its surroundings. The concept of neighborhood regarding relative proximity is expressed by the width of the spread function. This dispersion is fixed by the distance between each element and its nearest neighbor. Clusters are then formed in regions in which the sum of the partial functions reaches a threshold value.

The simplest case of clustering includes a set of only two dots. We can represent the dots (labeled p_1 and p_2) in one dimension with coordinates x_1 and x_2, respectively. The basic assumption is that two dots are always grouped because the distance, d, between exactly two dots can be related only to itself. That is, the clustering strength between only two dots is optimal no matter how large the distance is between those points. For the sake of simplicity, any restriction to the boundaries of the stimulus field is left out of consideration. In order to describe the clustering of p_1 and p_2, a normal distribution function, $f_{p_i}(x) = \exp - [(x_i - x)^2/2s_i^2]$, which is further defined as $[x_i, s_i]$, is superposed on each of the two dots $f_{p_1}(x) = [x_1, s_1]$ and $f_{p_2}(x) = [x_2, s_2]$, as is illustrated in Figure 2. The strength with which a dot influences its surroundings is reflected by the property of the normal function. The effect is maximal at its own position and monotonically decreasing to zero for its peripheral surroundings. The range of a point's influence is fixed by the standard deviation or dispersion, s. This parameter must be selected to yield an optimal representation of clustering strength. In relation to p_1 and p_2, it is clear that s_1 equals s_2 from the point of view of symmetry, so that $s_1 = s_2 = s$. When the dispersion is given a very small value with respect to the distance, d, between p_1 and p_2, the summed function $f_t(x) = [x_1, s] + [x_2, s]$ will show a minimum just between p_1 and p_2 (Figure 2a). In that case, $f_t(x)$ underrepresents the assumed strength of clustering. However, when the dispersion is large in comparison with d, the resultant function $f_t(x)$ will show a sharp peak between p_1 and p_2. In this case, $f_t(x)$ overrepresents the clustering of p_1 and p_2 (Figure 2b). It was decided, therefore, to take that value for s in which $f_t(x)$ just reaches a maximum, which emerges when $s = 1/2d$ (Figure 2c).

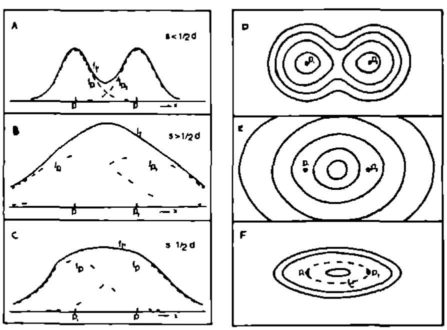

Figure 2. The functions $f_{p_1}(x)$, $f_{p_2}(x)$, and $f_t(x)$ are plotted as a function of x for $s < 1/2d$, $s > 1/2d$, and $s = 1/2d$, respectively (Figures 2a, 2b, and 2c). The two-dimensional representations are plotted in Figures 2d, 2e, and 2f, respectively. The set f_c of points ($\overline{x}$) for which it holds that $f_t(\overline{x}) = f_0$ is shown in Figure 2f.

The three clustering representations depicted in Figures 2a, 2b, and 2c are in one-dimensional form. Expanding them to a two-dimensional form yields a description of the clustering in terms of contours (see Figures 2d, 2e, and 2f), which are isolines of the two-dimensional function $f_t(\overline{x})$. In order to distinguish between the one-dimensional and two-dimensional situations, we now switch from scalar to vector notation. In this way, a contour is defined as that set of vector points ($\overline{x}$) for which it holds that $f_t(\overline{x})$ is a constant. We now come to the question of which constant should be selected. If a point is situated such that no clustering is possible (for instance, an isolated point), then this point virtually is an infinitely small cluster at its own position. This situation can be described by taking the threshold value to be the maximum height of the partial density function. So, the contour in question is now defined as that set of points ($\overline{x}$) for which it holds that $f_t(\overline{x}) = f_0$, with f_0 being the peak value of the spread function.

We now proceed to the case of a set of n dots, labeled $p_1, p_2, \ldots, p_n$, with positions $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$, respectively. A second assumption is made in these cases. That is, with the function $f_{p_i}(\overline{x})$ of a particular dot p_i in the set, the dispersion equals half the distance of that element to its nearest neighbor, $s_i = 1/2d(\overline{x}_i, \overline{x}_j)$, with $\overline{x}_j$ equaling the position of p_j, the nearest neighbor of p_i. In this way, the strength of a point's influence upon its surroundings depends on the distance to its nearest neighboring point. Once the dispersion value, and therefore the distribution function, has been established for each dot of the set, all partial distribution functions are summed:

$$f_t(\overline{x}) = \sum_{i=1}^{i=n} f_{p_i}(\overline{x}).$$

The resulting landscape of mountains and valleys can be seen to be quite similar to Marr's (Note 1) primary

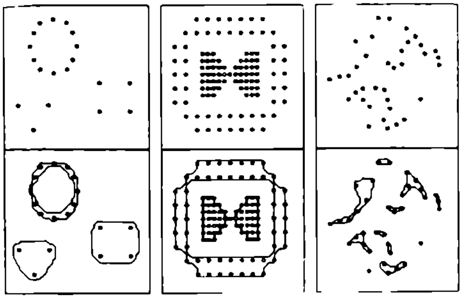

Figure 3 Three examples of dot figures that were subjected to CODE (top). Below are the resulting clusters within the dot patterns.

sketch The contour specified by the relation $f_t(\bar{x}) = f_0$ is computed along similar lines to the n = 2 case A cluster is then defined as the subset of all dots that lie precisely on, or within, that particular contour It is important to notice that clusters are formed in regions in which dots are mutually near neighbors rather than simply near neighbors CODE considers only relative proximity between dots, rather than absolute distances As a consequence, CODE is invariant under similarity transformations (translation, rotation, and changes of size) Also, CODE is insensitive to the order in which the points of a figure are scrutinized The proposed method is illustrated by a few examples in Figure 3

The following experiment examined whether CODE-determined clustering corresponded well with human observers' perceptions If CODE is a good predictor of human performance, it then makes sense to apply it in examining the influence of grouping on processing of visual numerosity

EXPERIMENT 1

Method

Subjects. Five undergraduate psychology students (four males one female), who were naive to the experimental task were paid for their participation in the experiment

Stimuli Seventy two dot figures were constructed Each figure differed both in number (n = 14 23) and arrangement of dots Seven different arrangements were used except for n = 22 and n = 23, for which there were eight arrangements

For each number of n, there was one configuration consisting of 1 large cluster (n_g = 1), one configuration of 2 clusters (n_g = 2), and so on, up to one configuration properly segmented into n_g = 7 (or n_g = 8 for n = 22 and n = 23) distinct clusters Care was taken that different clusters within one figure contain about the same number of dots Figures were constructed by assigning dots to different cell positions of a 15 by 15 matrix A dot was always placed randomly in the center of a cell matrix or just above, below, or next to that position for a distance of 2 of the cell width The objective criteria for the clustering within each stimulus were established by CODE Figure 4 depicts eight of the test stimuli with the same number of dots (n = 22) but with different arrangements

In addition to these test stimuli 18 fillers were constructed that depicted randomly arranged dot figures Ten of them (n = 14 23) were used to make the appearance of equal group sizes within one stimulus less obvious The other eight figures were used to avoid possible range effects Their numbers of dots were in the ranges 10 13 and 24 27

Procedure Participants were tested individually in a quiet laboratory room The stimuli were presented on a 27 x 20 cm video monitor situated approximately 75 cm from the participant and at eye level The dot diameter was 2 mm The size of the stimulus was made to fit as well as possible within an area of 10 x 10 cm with the midpoint of this area in the middle of the monitor Thus the stimuli were viewed at about an 8-deg angle Each stimulus was presented for a fixed duration of 100 msec Presentation order was random except that the first three stimuli were always filler stimuli The task was self paced a stimulus appeared on the screen 1 000 msec after the participant had pushed a button Participants were instructed to attend to the number of groups in each stimulus and to report this number orally The experimenter recorded the participants' responses Following 88 trials the participants rested a few minutes The next stimulus sequence contained the same stimuli presented in a different random order and in a different orientation Orientation was varied either by rotating the full configuration 90 deg or by mirroring along the vertical or horizontal axis All participants completed five or six series of 88 stimulus presentations each The experiment was run under the control of a PDP 11/45 computer

Results

Responses of all participants given with respect to a particular n_g value, regardless of the numerosity (n) of the stimulus, were classified into one category This categorization was done for each of the eight cluster levels (n_g = 1 through n_g = 8) Table 1 summarizes

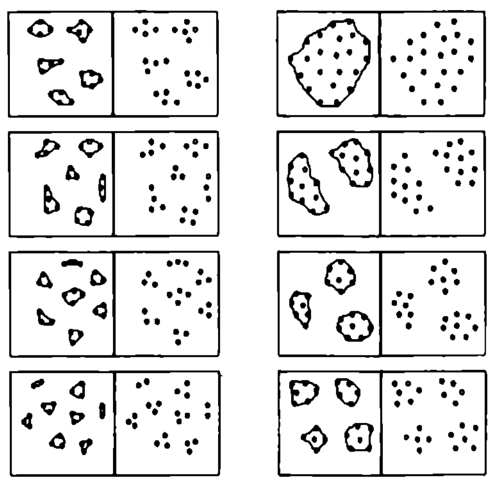

Figure 4 Eight test stimuli, with the same number of dots (n_d = 22), but differently arranged (n_g = 1-8)

Table 1
Reported Numbers of Groups (Subjective) vs. Presented Number of Clusters (Objective)

	R							
P	1	2	3	4	5	6	7	8
1	155	3	1	1	0	0	0	0
2	5	134	9	11	1	0	0	0
3	0	1	140	15	4	0	0	0
4	0	1	4	151	4	0	0	0
5	1	3	2	9	135	9	1	0
6	6	8	5	7	25	104	5	0
7	5	4	2	1	4	44	96	4
8	2	1	0	1	2	8	15	3

Note—R = reported number of groups, P = presented number of clusters

actually reported numbers of groups per n_g category. Figure 5 shows the percentages of responses that were in agreement with the CODE-predicted clusters for each category. The histogram shows high percentages (>80%) of CODE-predicted clusters for the levels $n_g = 1$ through $n_g = 5$. These percentages declined rapidly for the larger n_g values.

Discussion

The relationship between CODE-predicted clusters and humans' perceptions of groups was examined by asking participants to estimate the number of groups of dots in a stimulus. Participants were given only 100 msec to compose their estimates, a period that is evidently too short for successive one-by-one counting of groups. Hence, subitizing is the only means available to perform the task. As is well-known, subitizing has an upper limit of about five dots. This limitation is expressed dramatically in the data. For cluster values smaller than $n_g = 6$, the reported numbers of groups corresponded well with CODE predictions. The rapid decline in correspondence for larger n_g values most probably reflects the difficulty of subitizing these larger numbers. In order to overcome the problems of noisy results with larger n_g values, a latency experiment might seem a better experimental procedure. In this case, however, we cannot prevent participants' applying sophisticated cognitive strategies that might reflect more than proximity-based percepts.

From the given data, we conclude that CODE is robust enough to give a formal account of how grouping by relative proximity takes place in dot patterns.

In the next section, we will apply CODE to the analysis of how subjective grouping influences the processing of visual number of sets of dots.

Effects of grouping on the processing of visual number. Before describing how grouping affects the abstraction of number from figures with more than 10 dots, it is useful to distinguish between two grouping conditions. In one condition (called "small groups"), all distinct groups in a particular figure have about the same number of dots and these numbers are all within the subitizing range. An example is a configuration of 24 dots consisting of six groups of 4 dots each. In the second condition (labeled "large groups"), the various groups also contain the same number of dots, but their number now exceeds the subitizing span. This condition is exemplified by the case of 24 dots segmented into three groups of 8 dots each.

Response latencies to correct number responses provide an appropriate way to measure differential effects of groupability on the processing of visual number. The processing time for stimuli under the small-group condition is assumed to be composed of the following constituents: (1) a constant amount of time for the perceptual segmentation of a dot figure into groups, (2) time needed for subitizing the number of dots within a group, (3) time to compute the running sum of the partial results, and (4) a constant amount of time for the overt verbalization of the response. Thus, response latency is a function of both total number of dots, n_d, and the number of groups, n_g, and is expressed formally as

$$RT(n_d,n_g) = b_0 + n_d b_1 + (n_g - 1)b_2, \qquad (1)$$

in which b_0 stands for grouping and motor response time, b_1 stands for the time per dot consumed by the subitizing process, b_2 stands for the time to switch from one group to another and to compute the running sum, and RT stands for response latency. The number of these last operations equals $n_g - 1$. Since we are interested here in response latency as a function of group size (g_n), and since average group size ($\langle g_n \rangle$) equals n_d/n_g, Equation 1 can be rewritten as

$$RT(n_d,\langle g_n \rangle) = b_0 + n_d b_1 + [(n_d/\langle g_n \rangle) - 1]\, b_2 \qquad (2)$$

Dividing RT by n_d gives us the reaction time per single dot.

$$RT(n_d,\langle g_n \rangle)/n_d = b_1 + [(b_0 - b_2)/n_d] + (b_2/\langle g_n \rangle). \qquad (3)$$

The value of b_2 is about 400 msec (Vos & van Oeffelen, Note 3). The exposure duration in Experiment 1 was 100 msec, which was sufficient to allow perception of

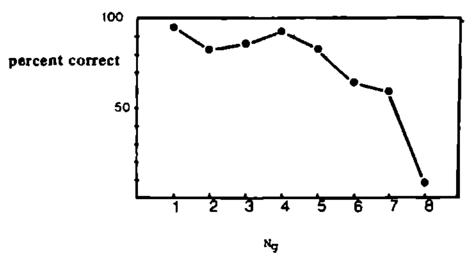

Figure 5. The percentages of responses that were in agreement with CODE-predicted clusters for each category n_g.

groups Eriksen and Collins (1968) reported a useful duration of dots in iconic memory of about 100 msec So, b_0 amounts to about 200 msec In addition to the fact that n_d is much larger than group size $\langle g_n \rangle$, we can ignore the second term in Equation 3 with respect to the third term and we may safely approximate Equation 3 by an expression for reaction time per dot that is a function of $\langle g_n \rangle$ only

$$RT(\langle g_n \rangle)/n_d = b_1' + (b_2/\langle g_n \rangle) \tag{4}$$

Equation 4 says that, under the small-group condition, the reaction time per dot is a hyperbolic function of average group size

We consider next the processing of dot figures in a large group Another quantification strategy has to be followed when the initial proximity-based groupings are too large to be subitized A plausible strategy is to first count one by one the dots within one group, after which the process of enumeration (see Beckwith & Restle, 1966) is continued with the dots of a second group, and so on The only advantage of groupability (large groups) might be that it facilitates the discrimination between dots already counted and those still to be counted Earlier studies on the processing of visual number (Aoki, 1977, Beckwith & Restle, 1966, Klahr, 1973) indicate that counting one by one requires about 300-400 msec for a dot to be enumerated, hence, response latency should be a linear function of n_d

$$RT(n_d) = A + Bn_d \tag{5}$$

However, the same studies also suggest that a subject might still form groups consisting of two, three, or perhaps even more dots during the process of counting If so, Equation 1, proposed for the case of small groups can also be applied to the present condition Nevertheless, an extra component has to be introduced representing the large proximity based groups The equation then reads as follows

$$RT(n_d,n_s) = b_0 + n_d b_1 + (n_s - 1)b_2 + b_3(n_d,n_s) \tag{6}$$

The number of subgroups is now represented by n_s, and $b_3(n_d,n_s)$ stands for the time to detect subgroups during the counting process Since we do not know how large the identified subgroups are and do not know, therefore, how many subgroups there are, we perform some averaging with respect to the grouping-dependent terms We assume that, averaged over a large set of dot figures under the large-group condition, a subject detects a fraction E(1) of ones, a fraction E(2) of twos, and so on, up to a fraction E(S) of subgroups the size of which equals the upper limit (S) of the span of subitizing In that case, the mean number of subgroups $\langle n_s \rangle$ should be

$$\sum_{i=1}^{i=S} E(i) \frac{n_d}{i} = \left\{\sum_{i=1}^{i=S} [E(i)/i]\right\} n_d$$

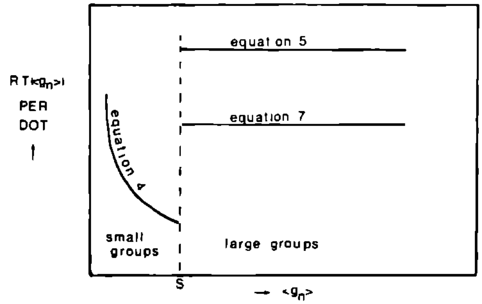

Figure 6 Expected reaction time per dot schematized as a function of initial (proximity-based) groupability $\langle g_n \rangle$ (number of dots within a group)

Further, we assume that the average time to perform such subgrouping increases proportionally with increasing total number of dots, n_d, so that $\langle b_3(n_d,n_s)\rangle = b_3' n_d$, of which b_3' is a constant With these assumptions made, we can rewrite Equation 6 as follows

$$\langle RT(n_d,n_s)\rangle =$$

$$b_0 + n_d b_1 + \left\{\sum_{i=S}^{i=S} [E(i)/i]\right\} (n_d - 1)b_2 + b_3' n_d \tag{7}$$

The factor

$$\left\{\sum_{i=1}^{i=S} [E(i)/i]\right\}$$

clearly is a constant, so all terms in Equation 7 are either constants or linear with respect to n_d Therefore, we can present Equation 7 in a much simpler form

$$\langle RT(n_d)\rangle = A' + B'n_d \tag{8}$$

Figure 6 schematizes the expected reaction time per dot as a function of initial (proximity based) groupability For dot figures with group sizes within the subitizing range, reaction time per dot is a hyperbolic function of average group size $\langle g_n \rangle$ For dot figures with larger group sizes, reaction time per dot should be independent of group size $\langle g_n \rangle$

EXPERIMENT 2

Method

Subjects. Eight undergraduate psychology students (seven males, one female) were paid for their participation in the experiment Five of them had also participated in Experiment 1

Stimuli The same set of stimuli used in Experiment 1 was also used in the present experiment

Procedure. The presentation of the stimuli differed from that of Experiment 1 in that each stimulus appeared on the monitor immediately after the participant had pushed a button The stimulus remained visible until the response, mediated by a microphone (Sennheiser headset), had surpassed a previously selected critical level Participants were asked to report the number of dots as quickly and accurately as possible Latencies were registered automatically The experimenter, who had a list of stimulus specifications, scored each response according to whether it was correct or incorrect and then stored it into the computer Whenever a participant committed an error, she/he received immediate feedback ("wrong") from the experimenter Each sequence of 88 stimuli contained the same stimuli presented in a different random order and in a different orientation As in Experiment 1, orientation was changed either by rotating the full configuration 90 deg or by mirroring along the vertical or horizontal axis

Each participant completed 12-16 sequences in three sessions spread out over 2 or 3 subsequent days Each session lasted about 90 min

Results

No more than 5% of participants' responses were incorrect The statistical analysis was restricted to correct responses given to the 72 test stimuli

The mean and standard deviation of a minimum of 12 and a maximum of 16 repeatedly measured latencies were computed for each subject and stimulus The standard deviations typically were on the order of 7%-12% and never reached values larger than 20% of the means The mean reaction times obtained over all participants are shown in Figure 7a as a function of n_d for each of the seven (or eight, for n_d = 22 and 23) different n_g conditions

Figure 7b shows the relation between reaction time and n_d in a more traditional way, in which averaged values over different n_g conditions belonging to the same n_d value are plotted The maximum and minimum values of response latency depicted in Figure 7a, regardless of groupability, are also plotted, as are reaction times on randomly arranged dot patterns derived from Jensen, Reese, and Reese (1950) Figure 7b shows that latencies roughly tended to increase with absolute number, regardless of dot arrangements

In order to minimize effects of interindividual differences in latencies, the averaged data of each subject were normalized Figure 8 presents the normalized mean reaction times as a function of mean group size $\langle g_n \rangle$ for each different value of n_d All 10 curves are patterned largely such that (1) for group sizes up to about five dots, latencies tended to decrease, (2) from $\langle g_n \rangle = 6$ onward, latencies rather abruptly jumped back to about the level obtained for $\langle g_n \rangle = 2$, and (3) for larger values of $\langle g_n \rangle$, there was a weak latency increase

Dividing the various latencies by the corresponding values of n_d, the overall means of processing time per dot were obtained as a function of $\langle g_n \rangle$ From value $\langle g_n \rangle = 2$ on, latencies per dot were subjected to a hyperbolic fit in accordance with the predictions formulated in Equation 1 Good fits were found for all latencies up to and including level $\langle g_n \rangle = 5$ [$RT/n_d = (1/\langle g_n \rangle \times 466) + 12$ msec/dot, p = 86] Taking the line of Equa-

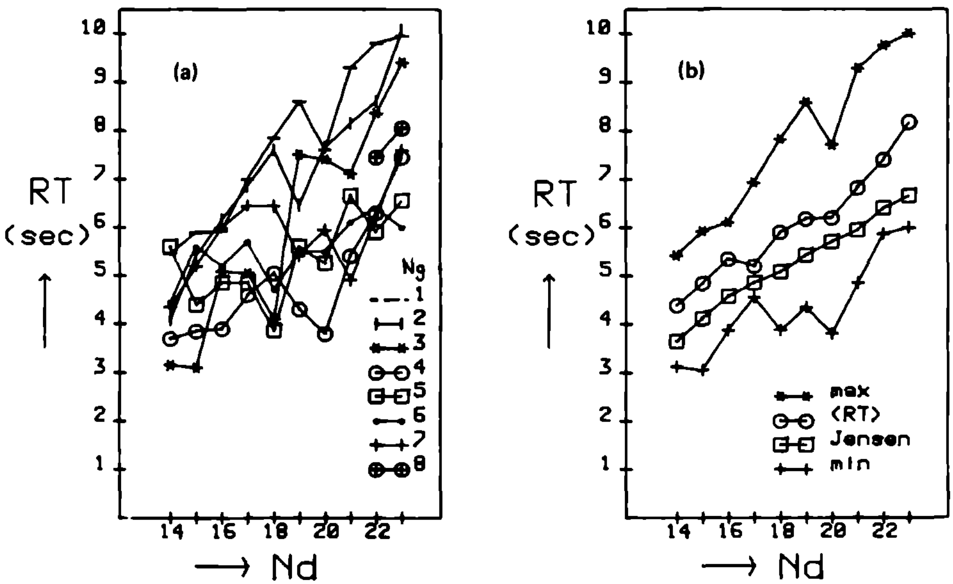

Figure 7. (a) For each of the seven (or eight, for n_d = 22 and 23) different n_g conditions (number of groups), the mean reaction times are plotted as a function of n_d, the total number of dots. (b) The mean reaction times over all n_g conditions are plotted as a function of n_d The maximum and minimum values of reaction time, depicted in Figure 7a, regardless of groupability, are also plotted, as are reaction times on randomly arranged dot patterns derived from Jensen et al. (1950).

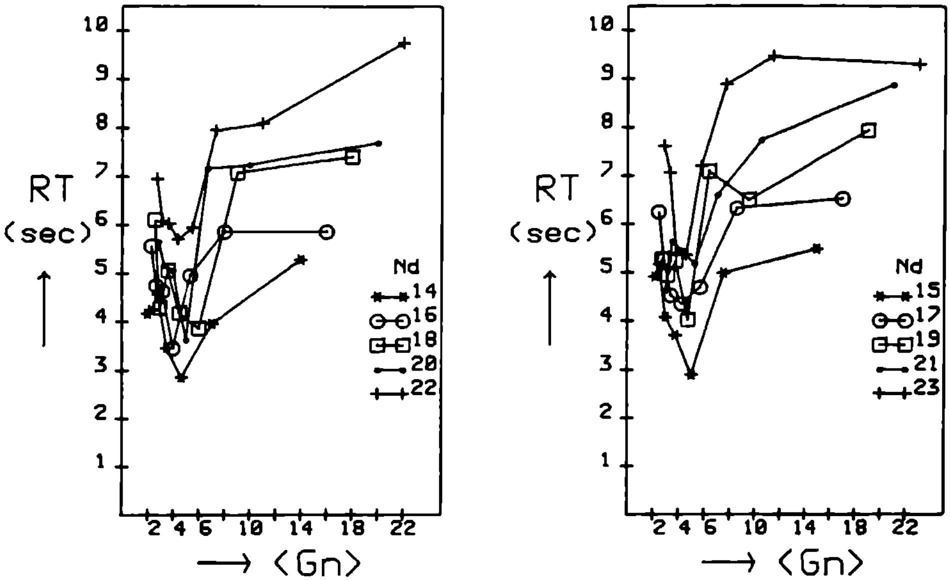

Figure 8. The normalized mean reaction times are plotted as a function of $\langle g_n \rangle$ for each number n_d.

tions 5 and 8, a linear fit was carried out on latencies at the higher $\langle g_n \rangle$ values to be reckoned from $\langle g_n \rangle = 23$ downward A good fit (p = 85) was found in the interval $6 \leqslant \langle g_n \rangle \leqslant 23$ ($RT/n_d = 4\langle g_n \rangle + 320$ msec/dot) The goodness of fit for the hyperbolic part, as well as for the linear one, deteriorated considerably when latencies in the interval $5 < \langle g_n \rangle < 6$ (shaded area in Figure 9) were also considered

Discussion and Conclusions

The data of Experiment 2 supported our theoretical description of how ascertaining the size of a large set of dots is affected by its patterning Latencies for one and the same set size appeared to follow two disjoint trends, the empirically derived transition point being the passage from (averaged) group size $\langle g_n \rangle = 5$ to $\langle g_n \rangle = 6$ Under the small-group conditions ($2 \leqslant \langle g_n \rangle \leqslant 5$), number was apparently ascertained by a combined strategy of subitizing the groups and adding the results to a running sum The hyperbolically shaped drop from approximately 350 msec/dot when $\langle g_n \rangle = 2$ to a minimum of roughly 200 msec/dot when $\langle g_n \rangle = 5$ confirmed our hypothesis that under small-group conditions, the processing time for the whole set decreases with a reduction in the number of adding operations The picture of the data obtained to the right of the transition point was more complex than predicted In accordance with the model, those data were appropriately fitted by a straight line The slope of the line, however, was not zero but slightly positive The increase of processing

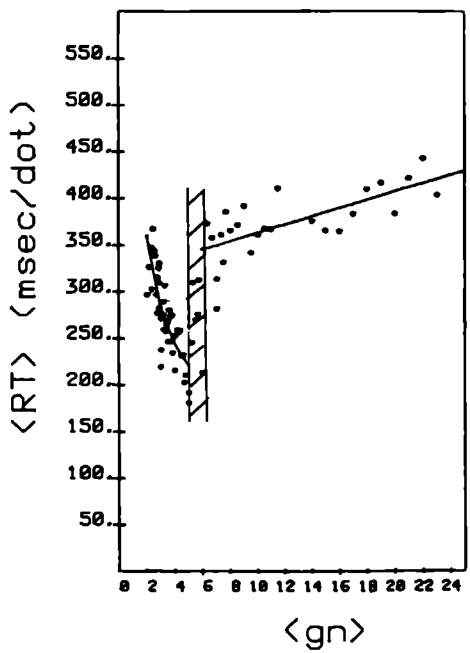

Figure 9 The normalized mean reaction times per dot are plotted as a function of $\langle g_n \rangle$

time per dot with larger $\langle g_n \rangle$ values was probably due to increased effects of lateral interference in progressively larger groups of dots, a factor not considered in our initial model Under the large-group conditions, the data suggested that the dots were counted in (sub)groups rather than one by one This conclusion is supported by the findings that mean processing time per dot was in the same order of magnitude in the $\langle g_n \rangle = 6$ to $\langle g_n \rangle = 10$ conditions as in the $\langle g_n \rangle = 2$ and $\langle g_n \rangle = 3$ conditions In other words, it seems that the perceiver splits the larger groups into subgroups of two or three dots (subjective grouping) and adds the results together in largely the same way as it is done under the small-group conditions. It is still unclear how subjective grouping takes place in the absence of proximity cues One possibility is that counting in twos and threes is a common routine for normal adults. For example, the chant "two, four, six, " is a slightly overlearned one for most of them. Another possibility is that the perceiver recognized patterns Neisser (1967) pointed to the fact that three randomly positioned dots nearly always make a triangle and it often is also easy to detect a quadrangle in four neighboring dots. Extending the search for meaningful patterns beyond clusters of three dots may increase, for example, the risk of lateral interference, with concomitantly greater errors This speculation warrants further research One might, for example, present large-group stimuli and ask the subject to ascertain the number of dots under different counting instructions, such as counting in twos, threes, fours, and so forth If our speculation is correct, latencies and errors should abruptly increase when counting must be done in fours or more

While the idea that the processing of visual number is affected by patterning of the objects is not new (cf Aoki, 1977, Atkinson et al , 1976a, 1976b, Beckwith & Restle, 1966, Bourdon, 1908; Freeman, 1912, Hamilton, 1865), our study is the first attempt to account for the number-pattern interaction on the basis of a quantitative theory Patterning was exclusively dealt with at its most elementary level of grouping by proximity Indeed, it is difficult to imagine even random configurations of dots that cannot be perceptually organized into distinct groups of relatively proximal dots. Consequently, the first step of our approach was to develop an algorithm to simulate how a human perceiver realizes the patterning in question. We think that CODE has been proved to be a useful tool for the quantitative specification of dot groupability in random patterns. Its applicability might be extended to other related problem fields, such as judgments of areas bounded by subjective contour

REFERENCE NOTES

1. Marr, D. *Analyzing natural images, a computational theory of vision* (AI Memo 334). Cambridge· M I.T. Artificial Intelligence Laboratory, 1975

2. Zucker, W., Rosenfeld, A., & Davis, L. S. *General purpose models: Expectations about the unexpected* (Technical Report 347). College Park: University of Maryland, Computer Center, 1975.

3. Vos, P. G , & van Oeffelen, M P. *Arithmetical operations with dot patterns. A chronometric analysis* (Internal Report ONO6). Nijmegen: University of Nijmegen, 1982.

REFERENCES

Aoki, T. On the counting process of patterned dots. *Tohuku Psychologica Folia,* 1977, **36**(1-4), 15-22.

Atkinson, J , Cambell, F. W., & Francis, M. R. The dependence of the visual numerosity limit on orientation, colour, and grouping in the stimulus. *Perception,* 1976, **5**, 335-342. (a)

Atkinson, J , Cambell, F W., & Francis, M. R. The magic number 4 ± 0: A new look at visual numerosity judgements. *Perception,* 1976, **5**, 327-334. (b)

Beckwith, M., & Restle, F Process of enumeration *Psychological Review,* 1966, **73**, 437-444.

Bourdon, B Sur le temps nécessaire pour nommer les nombres. *Revue Philosophique de la France et de l'Etranger,* 1908, **33**, 426-431.

Eriksen, C. W., & Collins, J. F Sensory traces versus the psychological moment in the temporal organization of form. *Journal of Experimental Psychology,* 1968, **77**, 376-382

Freeman, F. N. Grouped objects as a concrete basis for the number idea. *Elementary School Teacher,* 1912, **12**, 306-314.

Hamilton, W. *Lectures on metaphysics and logic* London: Blackwood, 1865.

Jensen, E. M., Reese, E. P , & Reese, T. W. The subitizing and counting of visually presented fields of dots. *Journal of Psychology,* 1950, **30**, 363-392

Kaufman, E. L., Lord, M. W., Reese, T. W., & Volkman, J. The discrimination of visual number. *American Journal of Psychology,* 1949, **62**, 498-525.

Klahr, D A production system for counting, subitizing, and adding In W G Chase (Ed), *Visual information processing.* New York: Academic Press, 1973

Klahr, D , & Wallace, J. G. *Cognitive development: An information processing view.* New York: Wiley, 1976.

Neisser, U. *Cognitive psychology* New York: Appleton-Century-Crofts, 1967.

Woodworth, R. S., & Schlosberg, H. *Experimental psychology.* New York· Holt, Rinehart & Winston, 1954

Zusne, L. *Visual perception of form.* New York: Academic Press, 1970.

(Received for publication November 5, 1981;
revision accepted February 10, 1982.)

Chapter 4

(Submitted for publication)

Enumeration of Dots:
An Eye Movement Analysis.

Michiel P. van Oeffelen and Peter G. Vos (*)

Abstract. The present study reports the measurement of response latencies and the recording of eye movements in a task where adults had to enumerate dots in figures which differed in number of dots (n_d=19-23) and grouping of dots. The functional relationship between latencies per dot and mean group-size was in agreement with earlier findings (van Oeffelen and Vos, 1982). Temporal information from eye movement data indicated that the contribution of fixation durations to overall latency was by far the largest. The contribution from saccades was considerably less though it superseded the contribution from eye blinks. Spatial information in the form of eye movement trajectories indicated that, in general, there occurred one or two fixations at the starting position. From this position onwards eye movements were directed towards areas of dots rather than to each dot in particular. Scanning behavior was sometimes reiterative in a sense that groups of dots were visited more than once. The results were discussed with respect to the nature of strategies employed during a dot enumeration task.

(*) The authors wish to thank L.H. Bavinck for his helpful comments on the manuscript.

In a previous study van Oeffelen and Vos (1982) chronometrically investigated the interactive effect of number of dots and their pattern upon the processing of visual numerosity. With dot figures of 14 through 23 dots it was found that the dots were not counted one-by-one but in groups of two or more dots. More precisely, when the number of dots within a proximity related (sub-)group of dots did not exceed five, the number was established by subitizing, a very rapid and accurate perceptual process (see Kaufman, e.a., 1949), the numerical result being transiently stored for further processing. The analysis of reaction times for those stimuli indeed showed that the main contributor to overall latencies was the time needed to sum up the various partial results of subitizing. However, when a stimulus field of dots could not be segmented into small groups on the base of proximity cues, the strategy of numerosity processing was not as clear. When discussing the data, the authors concluded that the most plausible strategy was counting in twos and threes rather than counting one-by-one. Sofar it remained unclear how subjective grouping of twos and threes took place in the absence of proximity cues.

One way to acquire an objective picture of the perceiver's strategy of subjective grouping and counting strategies based thereon is to analyse visual scanning patterns, which are considered as overt behavioral correlates of ongoing internal processes. Since it is the function of the eye to gather information, it seems reasonable to assume that it is generally directed toward regions of space that contain the most information. Thus, with respect to the counting process under study we expect relatively large saccades between subitizable groups of dots and only a few fixations located at their succesive positions while scanning trajectories are expected to show many fixations and small saccades for dot figures consisting of large groups of dots. In addition, eye movement trajectories could demonstrate whether subjects had restarted counting somewhere during the task. It was believed (Jensen, e.a, 1950; Klahr, 1973) that this latter phenomenon was responsible for the slightly positively accelerated function for reaction time versus number of dots.

The present study reports the measurement of response latencies and the recording of eye movements in a dot enumeration task. Eye movements were recorded using the pupil-center corneal-reflection method.

Method

Subjects. Seven undergraduate psychology students of the University of Nijmegen (five males and two females) were paid to participate in the experiment. All Subjects were naive with respect to the experimental task.

Stimuli. Thirty-seven dot figures were constructed. Each figure differed both in number (n_d=19-23) and arrangement of dots. Seven different arrangements were used, except for n_d=22 and n_d=23, for which there were eight arrangements. For each n_d, there was one configuration consisting of one large group, one configuration of two groups, and so on up to one configuration properly segmented into seven (or eight for n_d=22 and n_d=23) distinctly different groups. Care was taken that different groups within one configuration contained about the same number of dots. Objective criteria for grouping dots within a dot figure were established by CODE, a cluster algorithm which aimed to formalize the Gestalt rule of relative proximity (van Oeffelen and Vos, 1982, 1983). All dot figures were subjected to the algorithm yielding a description of their groupability in terms of contours around groups of dots. Figure 1 depicts seven of the stimuli with the same number of dots (n_d=21) but with different configurations. Code applied to the dot figures resulted into perceptually relevant boundaries (contours). These contours are

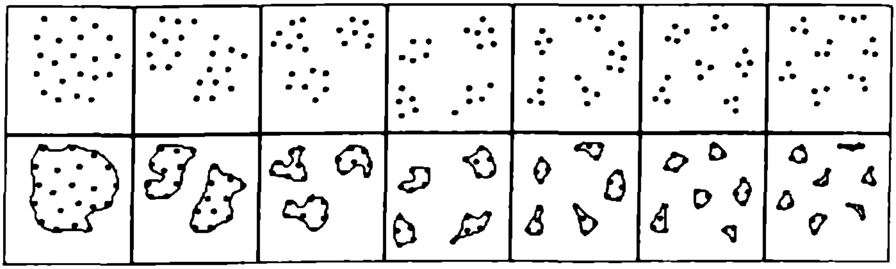

Figure 1: Seven of the stimuli with the same number of dots (n_d=21) but with different configurations. The perceptually relevant boundaries (contours) that were the result of CODE applied to the dots are also shown.

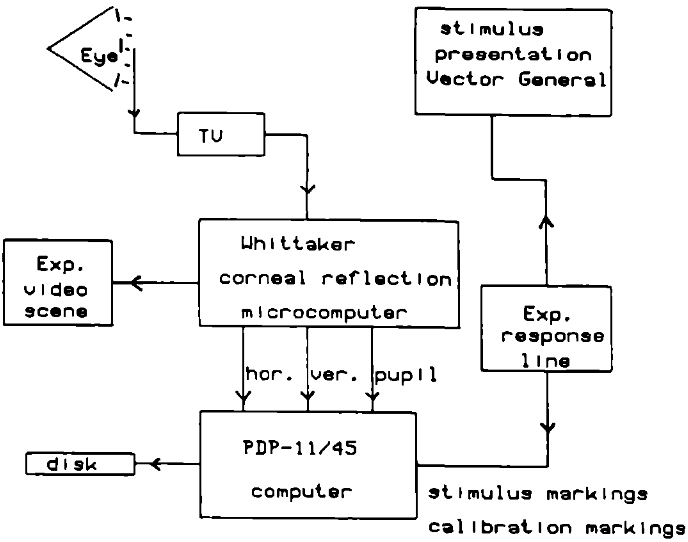

Figure 2: Schematic representation of the experimental set up.

also drawn in Figure 1.

Procedure. All Subjects were tested individually in a quiet laboratory room that contained the complete Whittaker 1998-S Eye View Monitor and TV-Pupillometer System (EVM), stimulus presentation screen and computer monitor, both connected to a PDP-11/45 computer system situated in a neighbouring room. Figure 2 shows a schematic representation of the experimental set up. The illumination in the room was dimmed during the actual experimental session. A Subject was seated in an adjustable chair and the head was held steady by a head-stand with backhead rests. The stimuli were presented on a 35*35 cm projection screen (Vector General). The Subject viewed the screen at eye level at approximately 1 meter (visual angle about 20^{0}) while a TV-camera photographed the Subject's left eye. This way, reflections were recorded from an infrared (IR) source light that was directed continuously at the eye. The IR light was filtered such as to absorb thermic radiation which could be harmful to the eye. The Subject's eye rotation and, consequently, his point of fixation was determined by measuring the center of the pupil with respect to the center of the corneal reflection. Because the center of the pupil and the center of the corneal reflection move together with eye rotation the difference between their positions was indicative of the eye's point of fixation. Thus, the eye position was independent of the head position as long as the pupil image was contained within the field of view of the TV-camera. The continuous flow of eye position information was presented as a spot superimposed on the video monitor scene available to the Experimenter. The digitalized output of the microcomputer was passed on to the PDP-11/45 computer. Eye position was calculated in terms of horizontal and vertical coordinates in the EVM-system representation. A third output was the pupil diameter measured in numbers of scan lines that intersected the image of the pupil on the Experimenter's TV-monitor screen. These three signals were delivered at a rate of 50 data points per second.

At the beginning of each experimental session a calibration procedure was started to match the EVM-coordinate system with that of the field of stimulus presentation. Calibration trials consisted of a Subject's fixating each position of a grid of nine calibration points. The nine points were situated in a 3*3 matrix such as to cover almost the entire presentation screen.

Once the calibration was carried out and calibration measures were stored on a data file together with Subject information, the experiment started. The Subjects had to fixate a point at the upper left corner of the screen before the presentation of each scene. Presentation order of the stimuli was random. The Subject was instructed to attend to the number of dots in each stimulus and to report this number orally. The task was self-paced; a stimulus appeared on the screen immediately after the participant had pushed a button. The stimulus remained visible until the response, mediated by a microphone (Sennheiser headset), had surpassed a previously selected critical level. Latencies were registered automatically. In the mean time, EVM data were gathered and stored on a data file. The Experimenter, who had a list of stimulus specifications, scored each response according to whether it was correct or incorrect and then stored it into the computer. Whenever a Subject committed an error, she/he received immediate feedback ('Wrong') from the Experimenter. Each Subject completed the session within half an hour.

Results

Not more than five percent of the 7 times 37 responses appeared to be errors and were discarded from further analysis.

To begin with, the mean and standard deviation of the measured latencies were computed for each stimulus and over all subjects. The standard deviations were in the order of 8-15 percent of the means. By dividing the mean latencies by the corresponding values of n_d, the overall means of processing time per dot were obtained. In Figure 3 are presented the mean latencies per dot ($\frac{\langle RT\rangle_{tot}}{n_d}$) as a function of mean group-size, $\langle G_n\rangle$. According to former theory (van Oeffelen and Vos, 1982), overall latencies per dot should satisfy a hyperbolic trend when mean group-size $\langle G_n\rangle$ does not exceed five dots:

$$(1) \quad \frac{\langle RT\rangle_{tot}}{n_d} = \frac{A}{\langle G_n\rangle} + B, \quad \langle G_n\rangle \leq 5, \quad A,\ B \text{ constants},$$

and should follow a linear trend for group-sizes larger than 6 dots:

$$(2) \quad \frac{\langle RT\rangle_{tot}}{n_d} = A'\langle G_n\rangle + B', \quad \text{for } \langle G_n\rangle \geq 6, \quad A',\ B' \text{ constants.}$$

The successive curve fittings applied to the mean latencies per dot yielded the following results:

$$\frac{\langle RT\rangle_{tot}}{n_d} = \frac{368.4}{\langle G_n\rangle} + 149.3, \quad r=.711, \quad \text{for } \langle G_n\rangle \leq 5,$$

and

$$\frac{\langle RT\rangle_{tot}}{n_d} = 3.4\ \langle G_n\rangle + 272.5, \quad r=.953, \quad \text{for } \langle G_n\rangle \geq 6.$$

The next step in the analysis of the experimental results concerned the eye movement recordings. To achieve visual scanning patterns in terms of fixations and saccades, the raw EVM data were subjected to a cluster algorithm. The algorithm was a slightly modified version from the one developed by Spaninks (1978). The algorithm yielded listings of positions and durations of fixations and saccades, and periods of disturbances (mainly eye blinks). Focusing the analysis upon the temporal information mean durations of fixations, $\langle RT\rangle_{fix}$, saccades, $\langle RT\rangle_{sac}$, and eye blinks, $\langle RT\rangle_{bli}$, were determined over all subjects. According to equations (1) and (2) curve fittings to the partial data were applied yielding the following results:

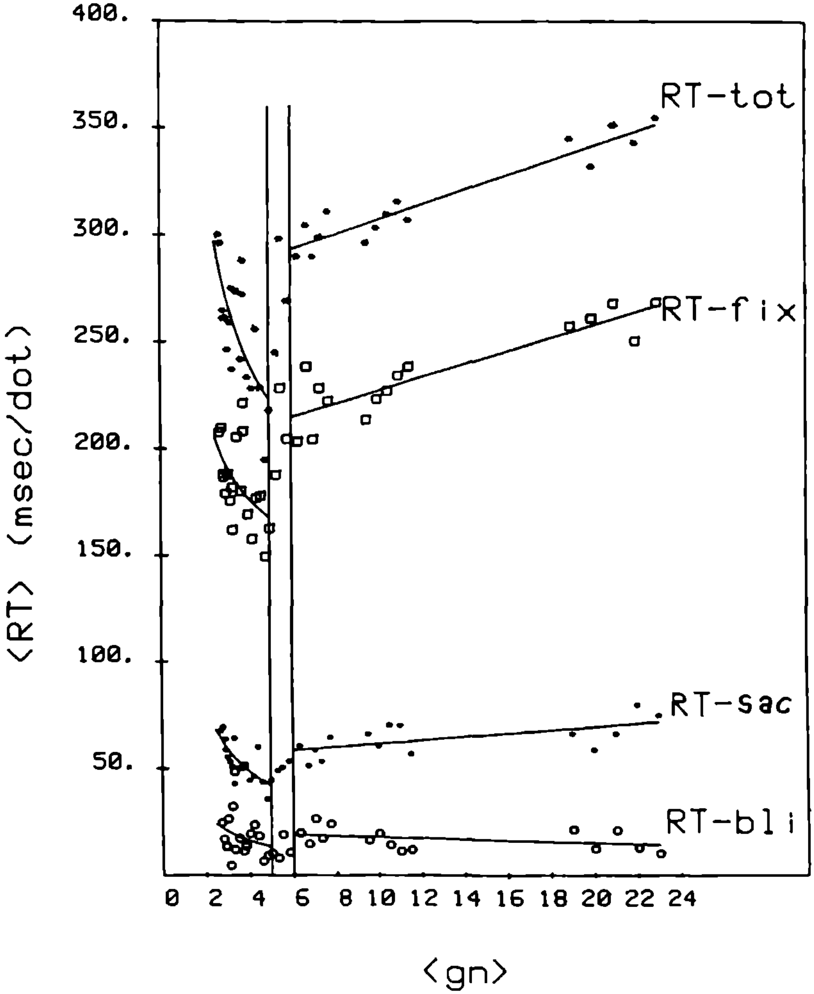

Figure 3: The mean latencies per dot, $\frac{\langle RT\rangle_{tot}}{n_d}$, and the mean contributions from fixations, $\frac{\langle RT\rangle_{fix}}{n_d}$, saccades, $\frac{\langle RT\rangle_{sac}}{n_d}$, and eye blinks, $\frac{\langle RT\rangle_{bli}}{n_d}$ are plotted as a function of mean group-size, $\langle G_n\rangle$. In addition, the best fitting hyperbolic curves for $\langle G_n\rangle \leq 5$ and best fitting straight lines for $\langle G_n\rangle \geq 6$ are drawn for both the overall results as well as the partial results.

fixations: $\frac{\langle RT\rangle_{fix}}{n_d} = \frac{185.5}{\langle G_n\rangle} + 130.6$, r=.500, for $\langle G_n\rangle \leq 5$,

$\frac{\langle RT\rangle_{fix}}{n_d} = 3.0\ \langle G_n\rangle + 196.5$, r=.888, for $\langle G_n\rangle \geq 6$.

saccades: $\frac{\langle RT\rangle_{sac}}{n_d} = \frac{128.7}{\langle G_n\rangle} + 16.2$, r=.744, for $\langle G_n\rangle \leq 5$,

$\frac{\langle RT\rangle_{sac}}{n_d} = .8\ \langle G_n\rangle + 54.0$, r=.614, for $\langle G_n\rangle \geq 6$.

eye blinks: $\frac{\langle RT\rangle_{bli}}{n_d} = \frac{54.2}{\langle G_n\rangle} + 2.4$, r=.281, for $\langle G_n\rangle \leq 5$,

$\frac{\langle RT\rangle_{bli}}{n_d} = -.3\ \langle G_n\rangle + 20.7$, r=-.358, for $\langle G_n\rangle \geq 5$.

From the fact that $RT_{tot} = RT_{fix} + RT_{sac} + RT_{bli}$ it is evident that, for both conditions of $\langle G_n\rangle$, addition of the partial fit results should satisfy the curve fitting to the overall results. In Figure 3 are plotted the mean latencies per dot, $\frac{\langle RT\rangle_{tot}}{n_d}$, and the mean contributions from fixations, $\frac{\langle RT\rangle_{fix}}{n_d}$, saccades, $\frac{\langle RT\rangle_{sac}}{n_d}$, and eye blinks, $\frac{\langle RT\rangle_{bli}}{n_d}$, all as a function of mean group-size, $\langle G_n\rangle$. In addition, the best fitting hyperbolic curves for $\langle G_n\rangle \leq 5$ and best fitting straight lines for $\langle G_n\rangle \geq 6$ are drawn for both the overall results, $\frac{\langle RT\rangle_{tot}}{n_d}$, and for the partial results, $\frac{\langle RT\rangle_{fix}}{n_d}$, $\frac{\langle RT\rangle_{sac}}{n_d}$, and $\frac{\langle RT\rangle_{bli}}{n_d}$. Finally, spatial information from the eye movement data was considered. Using the appropriate calibration parameters, positions of fixations and saccades in the EVM system representation were transformed into positions in the stimulus presentation system. Figure 4 represents some eye movement trajectories that were typically recorded during the counting task. Inspection of the trajectories revealed the following regularities. Corresponding to the prescribed starting position, the subject's eye position at the moment of stimulus onset was fixed at the upper left position of the stimulus field. Immediately after stimulus onset the eye did hardly move to

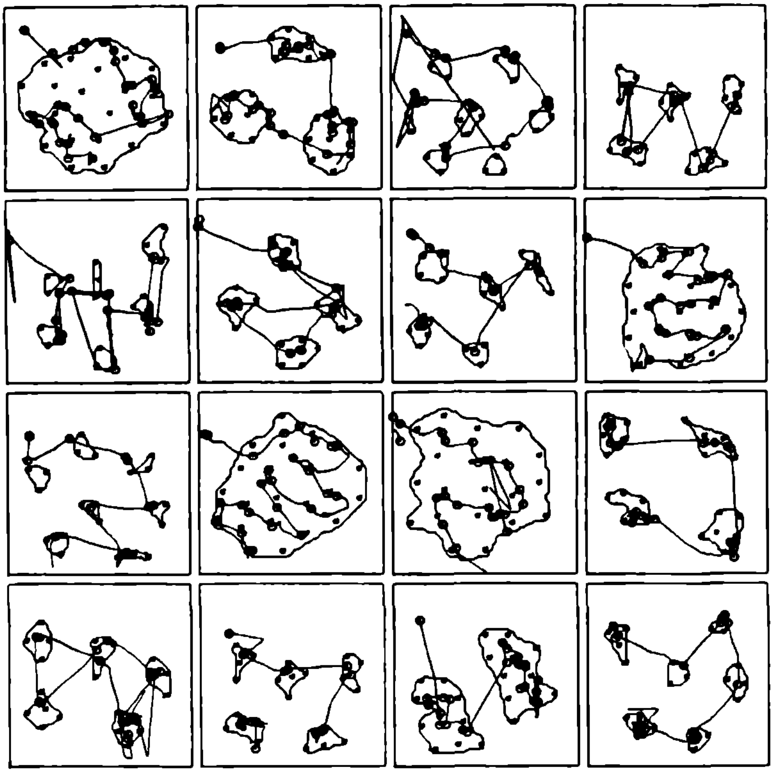

Figure 4: Some eye movement trajectories that were typically recorded during the counting task.

other positions; instead, one or two eye fixations were situated near the starting point. A mean duration of 305 ± 45 msec was found, a value which was independent from the number of dots and from the number of groups. From the starting position onwards eye movements were guided toward dots or groups of dots that were relatively closeby. Only in a few cases (6 percent) movements were directed to the midpoint of the stimulus figure. In general, it was found that scanning the figure occurred in a clockwise direction. Not only with small groups, $\langle G_n \rangle \leq 5$, but also with large groups fixations were directed to areas of dots rather than to individual dots. With small groups trajectories sometimes (11 percent) showed iterative scan behavior in a sense that an already scanned group of dots was revisited a second or even a third time. This behavior was not found with trajectories on large grouped stimuli.

Discussion

The chronometric analysis of the overall latencies yielded results that were in agreement with earlier findings (van Oeffelen and Vos, 1982): the functional relationship between latency per dot, $(\frac{\langle RT \rangle_{tot}}{n_d})$, and mean group-size, $(\langle G_n \rangle)$, appeared to be hyperbolic when $\langle G_n \rangle \leq 5$ and linear when $\langle G_n \rangle \geq 6$.

From Figure 3 it can be seen that the contribution of fixation durations to total latency exceeded that from durations of saccades while durations from eye blinks did hardly contribute to the overall latency. Under the condition of $\langle G_n \rangle \leq 5$ goodnesses of fit showed that durations from saccades did satisfy a hyperbolic trend better than did durations from fixations. The reason probably is that, during saccades, the cognitive operation of adding partial results can be carried out in parallel with the guidence of the eye movements, hence supporting a hyperbolic trend. Within a fixation duration extra time is demanded for to abstract information from the stimulus field and to preguide subsequent eye movements. The extra time might be responsible for disturbing a hyperbolic trend. A noisy occurrence of eye blinks probably is responsible for the low goodness of fit in question.

It is interesting to note that under the condition of $\langle G_n \rangle \geq 6$ a positive increase was found in $\frac{\langle RT \rangle_{fix}}{n}$ as a function of $\langle G_n \rangle$, while this increase was hardly the case with $\frac{\langle RT \rangle^{n_d}_{sac}}{n_d}$ and with $\frac{\langle RT \rangle_{bli}}{n_d}$ there was no increase at all. As already mentioned a positive increase was believed to be due to an increase in the number of restarts, but then, it should also occur with durations from saccades and from eye blinks. A more plausible explanation is that, with a larger number of dots, the duration of each fixation slightly increases. It is known (Mackworth and Bruner, 1970) that fixation duration partly is dependent upon the amount of information to be processed. Peripheral information progressively is needed with large numbers of dots to discriminate between what has been counted and what not. Hence, longer fixation durations are needed. Additional evidence against the proposition of an increase in the number of restarts comes from inspection of the eye movement trajectories. Neither with increasing n_d nor with an increase in $\langle G_n \rangle$ there was an increase in the number of restarts.

We now come to the question why there were one or two fixations needed before scanning could begin. Probably the Subject initially is structuring the stimulus field to inventory the task relevant problems. From a perceptual point of view, the Subject is building up a "primal sketch" (Marr, 1975) of a stimulus yielding a global picture of the arrangement of dots. From such a primal sketch a Subject could decide which proper strategy he has to follow. The question then arises why the initial fixations were not directed to the "center of gravity" of the dot figure. Such a result has been found (van Oeffelen and Vos, 1983) in an experiment where children of about five years old had to count a small number of dots. In 60 percent of the trials the children's eye movements were directed to the midpoint of the stimulus field immediately after stimulus onset. The difference in this respect between adults and children is probably due to the fact that adult's peripheral vision provides sufficient information to structure the visual field. Moreover, in order to minimize mnemonic difficulties due to lateral inference of what has been counted and what not it seems easier to start at a well defined begin-position (upper-left part of the screen). It could also explain why further counting occurs in a clockwise fashion.

It should be interesting to investigate whether the eye is directed to the centers of gravity of the various groups of dots. Will the eye jump to the centers of gravity of, for instance, a triangular or quadrulateral (sub-)figure? Some evidence for this has been found by Findlay (1980) using a scanning task with only two differently coloured spots to be scanned in one or the other prescribed order. He found that the eye moved as a rule to the midpoint between the two targets before performing the prescribed scanning path. Findlay interpreted the jumping to the midpoint as an indication of an initial, elementary processing of the global characteristics of the stimulus.

At this point some criticism must be ventilated regards the usefulness of the registration of eye movements. First, cognitive operations as part of a specific counting strategy can, in principle, be carried out with eyes closed. The continuous flow of eye movement information does not provide us with the precise moments of cognitive action. Second, records of eye movements can only show the succession of eye fixations, they cannot show precisely what information is being processed at each moment. Third, the point of fixation need not match the center of the field of attention. It is, for instance, well possible to shift attention to points in the periphery of the visual field without actually moving the eyes. All points of criticism are closely related and concern the problem of the status of attention. In addition to other methods such as the measurement of latencies and threshold values the registration of eye movements is one more method to bring this problem to a closer point of clarity.

References.

- Findlay, J.M. The visual stimulus for saccadic eye movements in human observers. Perception, 1980, 4, 361-377.

- Jensen, E.M., Reese, E.P., and Reese, T.W. The subitizing and counting of visually presented fields of dots. Journal of Psychology, 1950, 30, 363-392.

- Kaufman, E.L., Lord, M.W., Reese, T.W., and Volkman, J. The discrimination of visual number. American Journal of Psychology, 1949, 62, 498-525.

- Klahr, D. A production system for counting, subitizing, and adding. In W.G. Chase (Ed.), Visual Information Processing. New York: Academic Press, 1973.

- Mackworth, N.H., and Bruner, J.S. How adults and children search to recognize pictures. Human Development, 1970, 13, 149-177.

- Marr, D. Analysing natural images, a computational theory of vision. MIT AI Lab., AI Memo 334, 1975.

- van Oeffelen, M.P., and Vos, P.G. Configurational effects on the enumeration of dots: counting by groups. Memory and Cognition, 1982, 10 (4),396-404.

- van Oeffelen, M.P., and Vos, P.G. An algorithm for the description of relative proximity. _Pattern Recognition_, 1983, _16_, (3), 341-348.

- van Oeffelen, M.P. and Vos, P.G. The young child's processing of dot patterns: a chronometric and eye movement analysis. _International Journal of Behavioral Development_, 1983, _7_ (1).

- Spaninks, H. Analysis of eye movement recordings. Internal Report, Katholieke Hoge School, Tilburg, 1978.

(International Journal of Behavioral Development, vol.7 (1), 1984)

The Young Child's Processing of Visual Number: A Chronometric and Eye Movement Analysis.

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Abstract. The present study reports the measurement of response latencies and the recording of eye movements in a task where children of about 5.5 years had to count dot figures which differed in number of dots (n=1-8) and arrangement of dots. In agreement with earlier findings, response latencies for numbers of up to n=5 appeared to be indicative of the predominance of subitizing rather than counting strategies. Data from concomittant eye movement recordings clearly showed, that even the processing of the small numbers always required at least four fixations per response. Records of eye movements under the conditions of numbers of dots larger than n=5 were found to reflect mixed strategies and not elementary one-by-one counting procedures. Large processing times in comparison with adults were mainly due to interim verifications of results already established: children were, much more than adults, mentally loaded by the double task of storing partial results and processing new information at the same time.

(*) The authors wish to acknowledge Anny Bosman and Jan Wit for their assistance in the experiments and Michael Katzko for his helpful comments on the earlier draft of this paper.

In early studies on the processing of visual numerosity, it was generally assumed that human adults are able to use three major number-naming strategies: subitizing, counting, and estimating (see Woodworth and Schlosberg, 1954). The term ´subitizing´ was introduced by Kaufman, e.a. (1949) to indicate the process of immediate apprehension of a small number of visually presented objects. Counting was defined as the time-consuming method for the exact determination of larger numbers of objects. When not enough time is available to fulfill such a counting task, only estimates of the number of objects can be given, hence, estimation is referred to as global judgment of numerousness. Many of these studies were directed to the empirical question which strategy would be employed under what condition.

In a recent study, van Oeffelen and Vos (1982) have shown that subitizing can be understood in terms of probability concepts operating on subjective representations of numbers. It was assumed that the subjective representation is logarithmically related to the objective number representation. This implies, for instance, a subjective difference between 1 and 2 to be greater than that between 2 and 3, and so on up the scale. In accordance to psychophysical theories (Thurstone, 1928) it was argued that the classical Weber law, ds/s=constant, also holds for numbers. Experimentally, the Weber fraction for visual number was determined at 1/6. Consequently, a correct discrimination of more than 50 percent of the time is possible for the set of all pairs of numbers for which the difference of both divided by the minimal one does not exceed the critical Weber value. The set of numbers collapses to the classical span of apprehension when the difference between two neighbouring numbers is restricted to one. These are the small numbers with an upper limit of six, for six can be discriminated from seven above threshold but seven cannot from eight. Such a number discrimination hypothesis might make the process of estimation more comprehensible, though such factors as expectancy and arrangement can easily evoke systematic judgmental errors such as underesimation or overestimation (Birnbaum, 1975; Ginsburg, 1978; Indow and Ida, 1977; Krueger, 1982).

As to counting, van Oeffelen and Vos (1982) have studied the role of number-pattern interaction on the basis of a quantitative theory. It was found in experiments with stimuli consisting of large numbers of dots (n=14 through 23) that adults preferred counting by groups. Proximity based small groups consisting of up to 5 dots were found to be subitized, and the partial results summed to a running total. The operation of addition was responsible for high increases in reaction times. Proximity-based large groups were divided into smaller subgroups of two and three dots. It is, however, still unclear how this latter kind of grouping takes place in the absence of proximity cues. One possibility is that counting in twos and threes is a common routine for normal adults. For example, the chant 2,4,6,... is an overlearned one for most of them (Beckwith and Restle, 1966).

When it comes to counting a small set of objects, most children employ pointing and chanting, though some count silently and sometimes without explicite pointing. Beckwith and Restle (1966) mentioned the fact that children do show great sensitivity to the organization of the visual field, which suggests that grouping might also play a substantial role in children's counting. That is, even when a child is enumerating one-by-one, he may work rapidly and routinely within one group, pause and consolidate or 'store' his result in some way, and then attack the next group. Children may use spatial arrangement to get a simpler ordering of the stimulus to be quantified.

Young children accurately abstract the numerosity of small sets but rapidly lose accuracy as set size becomes greater than four or five (Beckwith and Restle, 1966; Svenson and Sjöberg, 1978). The fact that young children's number abstracting ability breaks down at about the point where adults appear to shift from subitizing to counting could be taken to imply that young children subitize rather than count. The perception of number, however, is not limited to the subitizing range. For instance, Smitsman (1982) has shown that even 6-year old children can be trained to give fairly accurate estimations of relative number. He also found that the estimations were significantly affected by the arrangement of the objects (squares and circles) into groups of variable size. If we say that a young child counts, we do not necessarily mean that the child follows the adult pattern. Adult's counting is fairly transparent: counting ability is governed by several principles and the successful ending of such a compound counting task requires the coordination of several component processes (van Oeffelen and Vos, 1982). With respect to this question, Fodor (1972) viewed the young child as processing a bundle of computational systems, formally analogous to those involved in adult cognition, but that are used for special purposes. Accordingly, cognitive differences between children and adults are quantitative rather than qualitative. Thus, the young child's use of the (computational) systems is tied to very specific situations. The adult's use of these same systems is not so restricted, and operating with them is extended to more and more domains.

As far as counting is concerned, children might possess the basic principles of counting (see Gelman, 1972; Gelman and Gallistel, 1978). It must be recognized that children apply some counting principles but not others; that individual principles may draw on component skills, some of which may not be perfected at a given age; and that some of the principles may operate more or less in isolation in the counting behavior of very young children. In the end, of course, successful counting involves the coordinated application of all the principles.

One way to investigate children's counting behavior is analyzing visual scanning patterns, considered as overt behavioral correlates of ongoing internal processes. Visual scanning can be viewed as a cognitively mediated process which reflects the individual's interests, his expectations about the visual environment, and his strategies for acquiring visual information. Moreover, since it is the function of the eye to gather information, it seems reasonable to suppose that it is generally directed toward regions of space that contain the most information.

Former methodology in the investigation of children's counting behavior mainly was concerned with the reaction time paradigm. Latencies were assumed to indicate the duration of the processing stages underlying number naming strategies, but they did not give a direct and complete picture of the strategies themselves. Studying scanning patterns can give answers to questions concerning strategies employed during a counting task. For example, a picture of the eye movement trajectories could indicate that children had counted one-by-one, or that they had counted by groups. The absence of any eye movements at all would indicate that they had subitized. Moreover, scanning patterns could learn us why children are so much slower than adults in performing a counting task.

Thusfar, data on developmental changes in eye movement patterns are still quite limited, perhaps because of the difficult and restricting nature of eye movement recordings and the time-consuming analysis required for them. Differences between children and adults in the speed, efficiency, systematicity and exhaustiveness of visual scanning have been summarized by Day (1975). Children do not fixate the most informative areas of a picture as frequently as adults do (Mackworth and Bruner, 1970) and young children do not use the same criteria for judging similarity among visual stimuli as older children (Vurpillot, 1968). The work of some Russian investigators (Zinchenko, e.a., 1963) demonstrated that children have a variety of scanning strategies at their disposal, but they do not always use the most appropriate

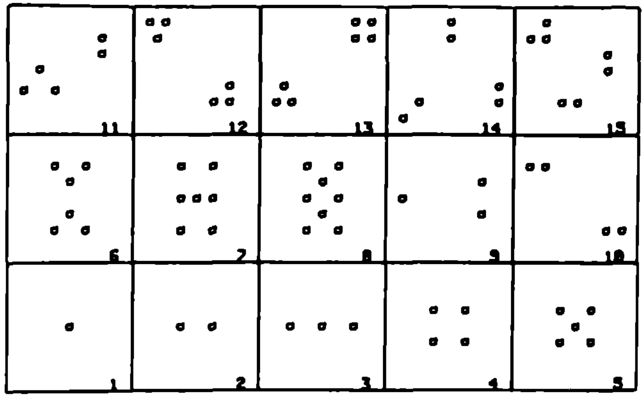


Figure 1: The fifteen stimuli that are used in the experiment. Each dot figure differed in number of dots (n=1-8) and/or arrangement of dots. The stimuli are divided into three categories; Category A (one group of dots): stimuli 1 through 8; Category B (2 groups): stimuli 9 through 13; Category C (3 groups): stimuli 14 and 15.

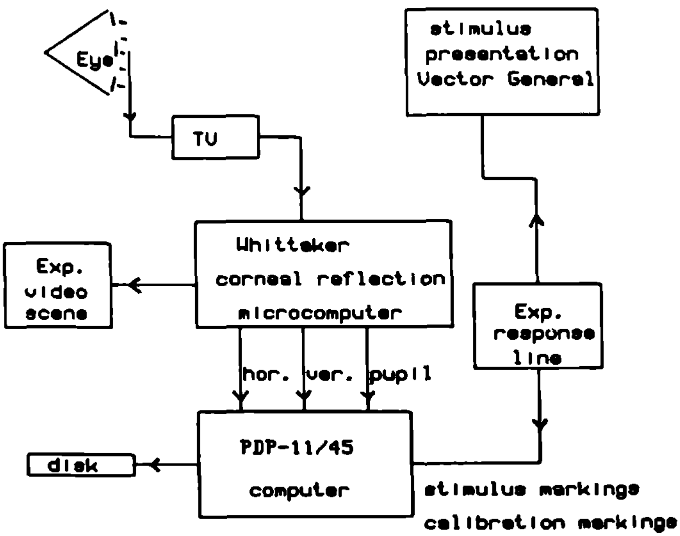


Figure 2: A schematic representation of the experimental situation.

or efficient strategy for a particular task.

The present study reports the measurement of response latencies and the recording of eye movements in simple number naming tasks with children of about 5.5 years of age. Eye movements were measured with the pupil-center corneal-reflection method.

Method

Subjects. Six kindergartners, three boys and three girls, from a nursery school associated with the psychological department at the University of Nijmegen, participated in the experiment. Their mean age was 5.6 years (range 5.3-5.9). Each child was given a picture-postcard in appreciation of her or his participation.

Stimuli. Fifteen dot figures were constructed. Each figure differed in number of dots (n=1-8) and/or arrangement of dots. The fifteen stimuli are depicted in Figure 1. It is convenient to divide the 15 stimuli into three categories. Category A consisted of all stimuli for which the dots were arranged into one group (stimuli 1 through 8). Category B-stimuli included all stimuli in which the dots were clustered into two groups (stimuli 9 through 13), while category C included the stimuli 14 and 15 in each of which the dots were arranged according to three groups of dots.

Procedure. The children were tested individually in a quiet laboratory room that contained the complete Whittaker 1998-S Eye View Monitor and TV-Pupillometer System (EVM), stimulus presentation screen and computer monitor, both connected to a PDP-11/45 computer system situated in a neighbouring room. Figure 2 shows a schematic representation of the experimental situation. The illumination in the room was dimmed during the actual experimental session. A child was seated in an adjustable chair and the head was held steady by a head-stand with backhead rests. In order to accomodate the child to the unusual environment, the Experimenter was throughout assisted by a female assistent known to the child. The stimuli were presented on a 35*35 cm projection screen (Vector General). The child viewed the presentation scene at approximately 1 meter and at eye level, while a TV-camera photographed the child´s left eye. This way, reflections were recorded from an infrared (IR) source light that was directed continuously to the eye. The IR light was filtered such as to absorb thermic radiation which could be harmful to the eye. The subject´s eye rotation, and, consequently, his point of fixation was determined by measuring the center of the pupil with respect to the center of the corneal reflection. The center of the pupil and the center of the corneal reflection move together with eye rotation; hence, the difference between their positions was indicative of the eye´s point of fixation. Thus, eye position was independent of head position as long as the pupil image was contained within the field of view of the TV-camera. The continuous flow of eye position information was presented as a spot superimposed on the video monitor scene available to the Experimenter. The digitalized output of the microcomputer was passed on to the PDP-11/45 computer. Eye position was calculated in terms of horizontal and vertical coordinates in the EVM-system representation. A third output was the pupil diameter measured in numbers of scan lines that intersected the image of the pupil on the Experimenter´s TV-monitor screen. These three signals were delivered at a rate of 50 data points per second.

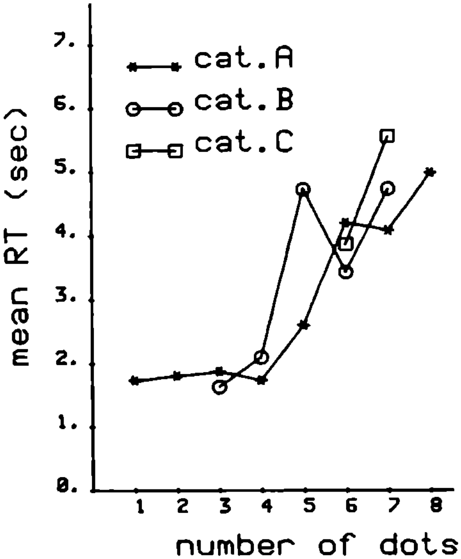
At the beginning of each experimental session a calibration procedure was started to match the EVM-coordinate system with that of the field of stimulus presentation. Calibration trials consisted of a subject´s fixating each position of a grid of nine calibration points. The nine points were situated in a 3*3 matrix such as to cover almost the entire presentation scene. To fixate a child's attention to a particular point, a small triangle, a circle and a square were alternatively presented in the adequate position with a rate of one figure per second, and the child had to count aloud one of these figures.

Once the calibration was carried out and calibration measures were stored on a data file together with subject information, the experiment started. The children had to fixate a point at the upper left corner of the screen before the presentation of each scene. Presentation order of the fifteen stimuli was random. The child was instructed to attent to the number of dots in each stimulus and to report this number orally. The Experimenter recorded the child's responses. Response latencies were recorded from the onset of the stimulus presentation till the experimenter stopped the timer immediately after the response had been given. In the mean time, EVM data were gathered and stored on a data file. Each child completed the session within half an hour.

Results

First, the chronometrical findings will be dealt with after which the eye movement data are subjected to the analysis. Not more than five percent of the 6 times 15 responses appeared to be errors which were discarded from further analysis.

To begin with, the mean and standard deviation of the measured latencies were computed for each stimulus and over all children. Figure 3a shows the relation between mean reaction time (RT) and number of dots for the three categories A, B, and C. Figure 3b indicates the standard deviations (SD) for each of the experimental conditions. From Figures 3a and 3b, it can be seen that in category A, stimuli 1 through 4 all resulted in nearly equal and low



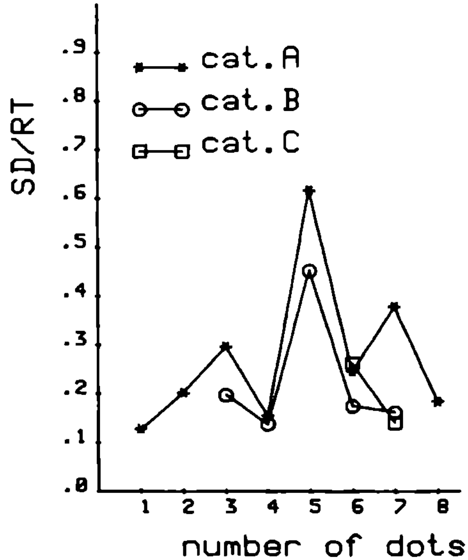
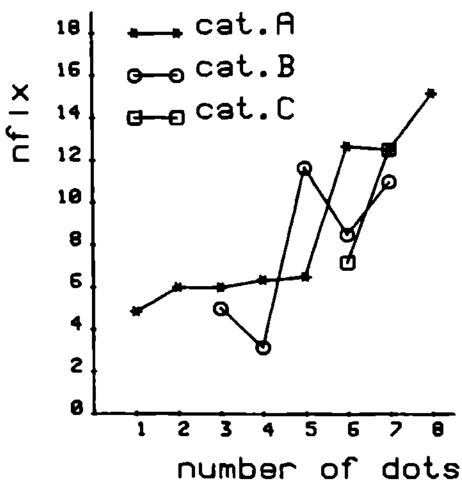
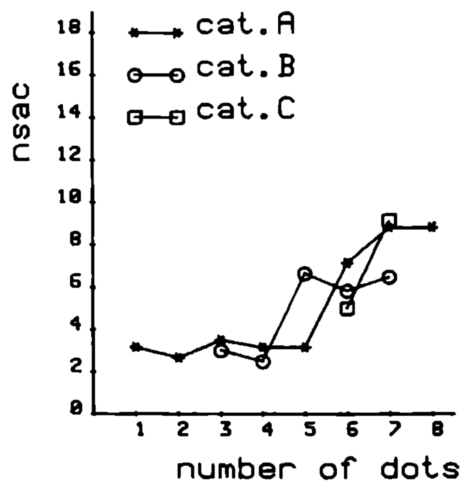


Figure 3a: The relation between mean reaction time (RT) and number of dots for the three categories A, B, and C.

Figure 3b: The relation between standard deviation (SD), relative with respect to RT, and number of dots for both categories.





Figures 4a and 4b: The mean number of fixations (nfix) and the mean number of saccades (nsac) on each stimulus and over all children plotted as a function of the number of dots.

RT- and SD-values. It is tentatively concluded that the dots of these configurations were subitized by all children. For 5 dots onwards, RT's and SD's increased considerably. However, the fact that the SD for stimulus 5 was rather high with respect to its moderate mean RT-value suggests that some children subitized this number of dots while others used a more time-consuming strategy. The strategy for number naming in the case of stimuli 6,7, and 8 was definitely not subitizing as is evident from inspection of the corresponding RT-values.

Category B-stimuli 9 (3 dots) and 10 (4 dots) had about the same RT- and SD-values as the corresponding category A-stimuli. As for the A-stimuli we now conclude that the dots of stimuli 9 and 10 were subitized. The RT- and SD-values for the remaining B-stimuli 11 through 13 and C-stimuli 14 and 15 all were of the same order of magnitude as the A-stimuli 6, 7, and 8.

The next step in the analysis of the experimental results concerned eye movement recordings. The EVM system delivered digital information in terms of horizontal and vertical EVM coordinates measured per 20 msec, as well as pupil diameter information. To achieve visual scanning patterns in terms of fixations and saccades, the raw EVM data were subjected to a cluster algorithm. The algorithm was a slightly modified version from the one developed by H. Spaninks (1978). The algorithm yielded listings of fixations, saccades, and periods of disturbances (mainly eye blinks). Finally, using the appropriate calibration parameters, positions of fixations and saccades in the EVM system representation were transformed into positions in the stimulus presentation system.

For each of the three categories A, B, and C the mean number of fixations (nfix) and mean number of saccades (nsac) on each stimulus and over all children are plotted in Figures 4a and 4b as a function of the numbers of dots. Generally, the number of fixations exceeded that of the saccades, a finding

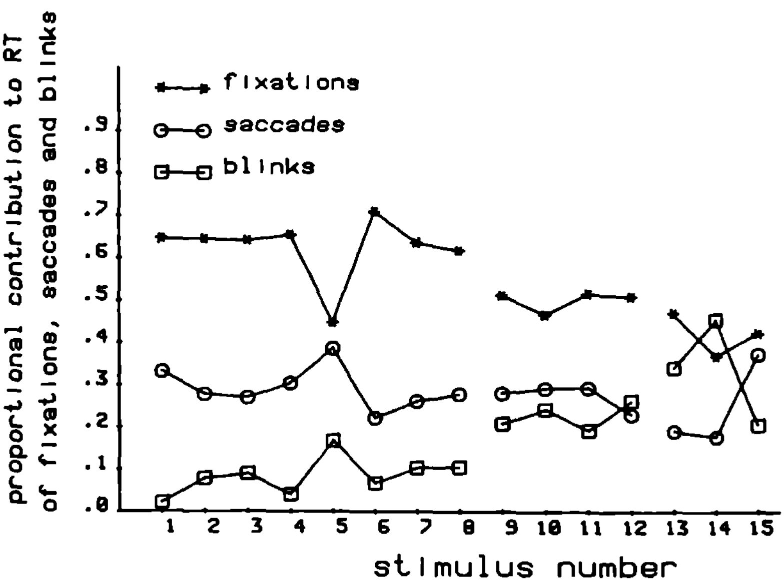


Figure 5: Proportional contributions to RT of durations of fixations, saccades, and eye blinks.

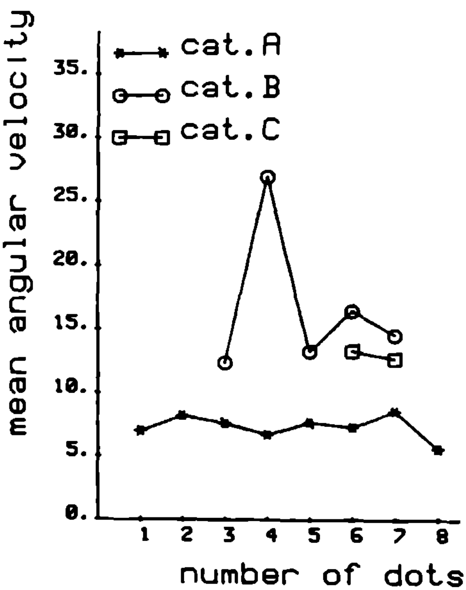


Figure 6: The mean number of angular velocities plotted as a function of the number of dots for each category.

which emphasizes the fact that two fixations might follow each other very close in time. It should be observed that even for the A-stimuli with less than five dots, the number of fixations was always larger than one, which is hard to reconcile with the idea that subitizing is the perception of number at one glance. We come back to this point in the final section.

Figure 5 illustrates how total mean RT is composed of contributions from durations of fixations, saccades, and eye blinks. As can be seen from Figure 5, fixation duration was the major contribuant to RT for cat. A-stimuli. Saccadic movements affected RT only moderately whereas eye blinks had little influence. The contribution of eye blinks increased considerably for the B-stimuli and the effect was even stronger for the C-stimuli. As a consequence of the increased contribution of eye blinks to RT the contribution of fixations decreased proportionally. It seems reasonable to suppose that the increased importance of eye blinks to B- and C-stimuli followed from the fact that the informative loci in terms of groups of dots were considerably more dispersed over the visual scene, whence larger trajectories of the eye movements were demanded for, coupled with automatically closing the eyes. Another explanation is that when children counted the dots within a group, they had to add this number to one already stored in a working memory. As a result of overload children might be confused and react by closing the eyes for a short while.

One might expect that large saccadic movements to the B- and C-stimuli consequently imply longer saccades, hence a larger contribution to RT. Such a consequence is questionable as we can see in Figure 5. Moreover, large saccadic movements can be compensated for by a proportional increase of angular velocities of the eye. That the latter phenomenon is likely to occur can be observed in Figure 6, in which for each category the mean angular velocities are plotted as a function of the number of dots.

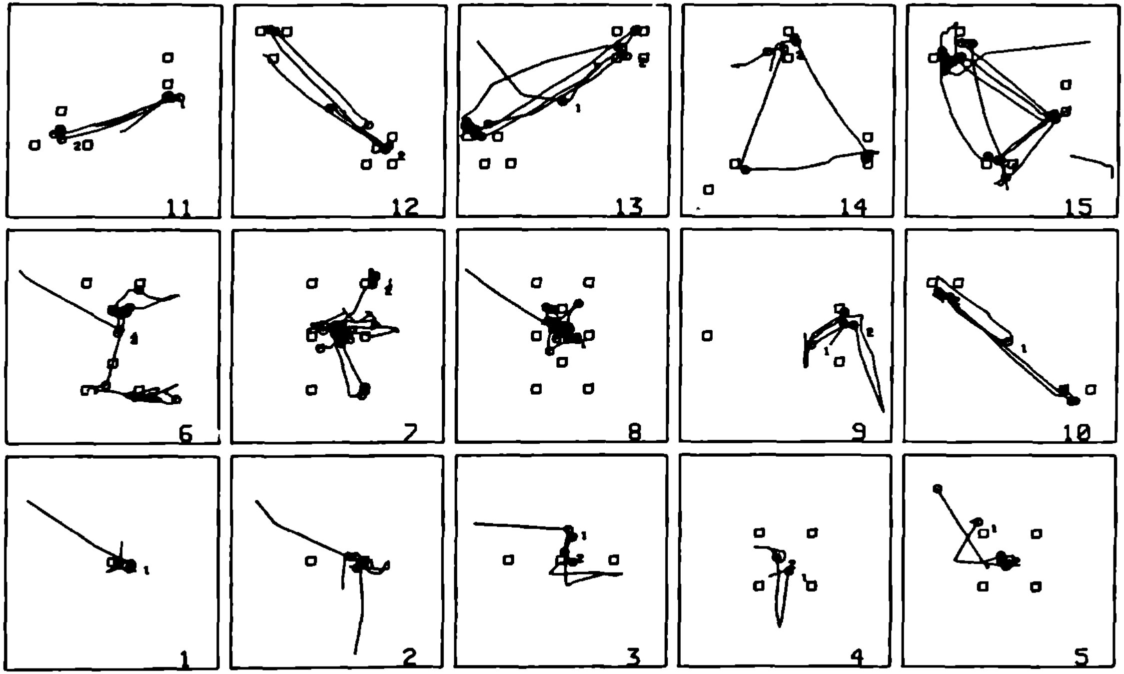


Figure 7: The scanning patterns as they were measured for the different stimuli, obtained from one child which was representative for the group of children. The first and second fixation points were labeled with numbers 1 and 2 to indicate the starting position of the trajectories.

Finally, scanning patterns as they were projected onto the corresponding stimulus fields were analyzed for each child. Figure 7 shows the scanning patterns as they were measured for the fifteen stimuli, obtained from one child which was representative for the group. An inspection of the scanning patterns revealed the following regularities. First, the starting point of the various patterns were in 60 percent of the cases located in, or near the midpoint of the stimulus field, irrespective of the presence or absence of a dot at that position. The inclination to start at the midpoint may indicate the tendency to process the stimulus with respect to its wholistic features before the proper number processing task is executed. Second, scanning patterns rarely implied a child's counting dot-by-dot. Such a possibility is by no means excluded, however, since scanning may, at least partly, be done parafoveally. Third, especially for stimuli with groups of unequal size (stimuli 11, 13, and 15), scanning from group to group was reiterative, in a sense that a group which was already scanned was revisited a second or even a third time. The behavior is indicative of difficulties with storing partial results in a memory buffer while processing other information.

Discussion

The RT-data for the stimuli 1 (n=1) through 8 (n=8) were in agreement with those of earlier studies (Beckwith and Restle, 1966; Klahr and Wallace, 1976), according to which children around five years of age subitize numbers of up to about five dots, while larger numbers are processed by one or another method of sequential counting. The fact that the RT's to stimuli 1 through 4 were roughly three times larger than those typically reported for adults must mainly be attributed to the difficulties with the retrieval of the appropriate number labels by children (see Klahr and Wallace, 1976). This explains why none of the concomittant EVM-data consisted of scanning patterns with less than four fixations. This finding is important from a developmental point of view, since it has been found that for adults only one fixation is the rule rather than an exception with those stimuli (Chi and Klahr, 1975). Consequently, subitizing in the sense of perceiving numerosity at one glance does not yet occur in five year old children.

Another point of developmental interest concerns the finding that none of the EVM-data for stimuli with more than five dots unambiguously indicated the children's use of an exhaustive dot-for-dot scanning. In other words, the present data do not permit us to conclude that the children counted one-by-one. More likely some mixed strategies were predominantly followed, consisting of subitizing dots occuring in small groups and adding these results to a running total. The children behaved in the same way as adults do (van Oeffelen and Vos, 1982), which supports the earlier mentioned viewpoint of Fodor (1972) regarding the limits of the young child's cognitive abilities being quantitative rather than qualitative in nature.

In order to follow mixed strategies, the children must have been able to make a 'primal sketch' (Marr, 1975) of a stimulus, yielding a global picture of the arrangement of the dots. Such a primal sketch purportedly takes place within about the first 300 milliseconds (Marr, 1975, van Oeffelen and Vos, in press). Its presence at the very beginning of the children's EVM-trajectories is supported by the observation that in nearly 60 percent of all cases the first one or two fixations centered around the midpoint of the stimulus field. Similar results have been found by Findlay (1980) using a scanning task with only two differently coloured spots, to be scanned in one or the other prescribed order. He found that the eye moved as a rule to the midpoint between the two targets before performing the prescribed scanning path. Findlay interpreted the jumping to the midpoint as indicative of an initial, elementary processing of the global characteristics of the stimulus.

A third point of developmental relevance is concerned with why the response latencies of the children to the stimuli with more than five dots were also much longer than those reported for adults (Beckwith and Restle, 1966; Klahr, 1973). The EVM-data allow us to suppose that the children spent much more time than adults on interim verifications of results already established: they are, much more than adults, mentally loaded by the double task of storing partial results and processing new information at the same time.

Children around tha age of five years progressively learn, as a part of their kindergarten education, to attach number labels to collections of up to about ten objects. Although there exists a massive corpus of developmental studies on the origin of the child's capacities in this respect, only a few of them were addressed to the question of how preschoolers proceed in establishing the cardinal number of a small collection (see Gelman and Gallistel, 1978). The reason probably has to do with Piaget's influential emphasis on the study of the young child's tendency to make nonconservatory responses when asked to state ordinal judgments ('more' or 'less') regarding collections differing only in the spatial arrangement of a same number of objects. It is clear, however, that even under the Piagetian conditions of comparing two quantities, the time it takes to process each of them as well as the quantification strategies used by the child, are valuable sources of information about the child's perceptual and cognitive capacities. The kind of information obtained from the data of the present study may contribute to the rather un-Piagetian idea that mental development is a matter of subtle, quantitative progress rather than a movement through qualitatively different stages.

References.

- Beckwith, M., and Restle, F. Process of enumeration. Psychological Review, 1966, 73, 437-444.

- Birnbaum, M.H. Expectancy and judgment. In F. Restle, R. Shiffrin, N. Castellan, H. Lindman, and D. Pisoni (Eds.), Cognitive Theory, Volume 1, Hillsdale, N.J.: Erlbaum, 1975.

- Chi, M.T.H., and Klahr, D. Span and rate of apprehension in children and adults. Journal of Experimental Child Psychology, 1975, 19, 434-439.

- Day, M.C. Developmental trends in visual scanning. In H.W. Reese (Ed.), Advances in Child Development and Behavior, (Volume 10). New York: Academic Press, 1975.

- Findlay, J.M. The visual stimulus for saccadic eye movements in human observers. Perception, 1980, 4, 361-377.

- Fodor, J.A. Some reflections on L.S. Vygotky's 'Thought and Language'. Cognition, 1972, 1, 83-95.

- Gelman, R. The nature and development of early number concepts. In H.W. Reese (Ed.), Advances in Child Development and Behavior, Volume 7. New York: Academic Press, 1972.

- Gelman, R., and Gallistel, C.R. The child's understanding of number. Cambridge, Massachusetts: University Press, 1978.

- Ginsburg, N. Perceived numerosity, item arrangement, and expectancy. American Journal of Psychology, 1978, 91 (2), 267-273.

- Indow, T., and Ida, M. Scaling of dot numerosity. Perception and

Psychophysics, 1977, 22, 265-276.

- Kaufman, E.L., Lord, M.W., Reese, T.W., and Volkman, J. The discrimination of visual number. American Journal of Psychology, 1949, 62, 498-525.

- Klahr, D. A production system for counting, subitizing, and adding. In W.G. Chase (Ed.), Visual Information Processing. New York: Academic Press, 1973.

- Klahr, D. and Wallace, J.G. Cognitive Development: An information Processing View. New York: Wiley, 1976.

- Krueger, L.E. Single judgments of numerosity. Perception and Psychophysics, 1982, 31 (2), 175-182.

- Mackworth, N.H., and Bruner, J.S. How adults and children search and recognize pictures. Human Development, 1970, 13, 149-177.

- Marr, D. Analyzing natural images, a computational theory of vision, MIT AI Lab., AI Memo, 334, 1975.

- van Oeffelen ,M.P., and Vos, P.G. Configurational effects on the enumeration of dots. Memory and Cognition, 1982, 10 (4), 396-404.

- van Oeffelen, M.P., and Vos, P.G. A probabilistic model for the discrimination of visual number. Perception and Psychophysics. 1982, 32 (2), 163-170.

- van Oeffelen, M.P., and Vos, P.G. An algorithm for pattern description on the level of relative proximity. Pattern Recognition. 1983, 16 (3), 341-348.

- Smitsman, A.W. Perception of number. International Journal of Behavioral

Development. 1982, 5, 1-31.

- Spaninks, H. Analysis of eye movement recordings. Internal Report, Katholieke Hoge School, Tilburg, 1978.

- Svenson, O., and Sjöberg, K. Subitizing and counting processes in young children. Scandinavian Journal of Psychology, 1978, 19, 247-250.

- Thurstone, L.L. A law of comparitive judgment. Psychological Review, 1927, 34, 273-286.

- Vurpillot, E. The development of scanning strategies and their relation to visual differentiation. Journal of Experimental Child Psychology, 1968, 6, 632-656.

- Woodworth, R.S., and Schlosberg, H. Experimental Psychology. New York: Holt, Rinehart and Wilson, 1954.

- Zinchenko, V.R., Chzhi-Tsin, V., and Tarakanov, V.V. The formation and development of perceptual activity. Soviet Psychology and Psychiatry, 1963, 2, 3-12.

Slotbeschouwing.

Zoals we in de inleiding van dit proefschrift konden lezen werd het eerste feitelijke aantalbenoemingsexperiment uitgevoerd door Jevons in 1871. Bij het zeer snel schatten van het aantal bonen welke, na er een handvol van opgegooid te hebben, op een schoteltje terecht kwam, viel het hem op dat hij zelfs bij een aantal van vijf vergissingen maakte. Het aantal foute schattingen nam bij grotere aantallen alleen maar toe. Laten we Jevons experiment eens wat uitbreiden. Stel er zijn nu twee schoteltjes, op het ene liggen 20 bonen, op het andere 21. Het zal duidelijk zijn dat het moeilijk zoniet onmogelijk is om heel snel en met zekerheid te zeggen waar er 20 bonen en waar er 21 liggen. Uit Hoofdstuk 1 blijkt dat, wanneer het aantal van 20 bonen vervangen wordt door een aantal van 18, gemiddeld genomen zo'n 50 procent van de tijd een correct onderscheid gemaakt kan worden tussen 18 en 21. Hetzelfde nu geldt voor aantallen 12 en 14 waarbij het verschil 2 is en ook voor aantallen 6 en 7 met een verschil van 1. Deze voorstelling sluit precies aan bij de suggestie, geopperd door Averbach (1963) dat subiteren opgevat zou kunnen worden als een vorm van visuele discriminatie. De bovengrens van de subiteerspanne is dan een soort van Weber-drempel voor visueel aantal. De experimentele resultaten ter toetsing van het waarschijnlijkheidsmodel voor de discriminatie van visueel aantal zoals gepresenteerd in Hoofdstuk 1 bevestigden Averbach's hypothese. Sterker nog, het blijkt dat de klassieke subiteerspanne, bestaande uit de opeenvolgende aantallen 1, 2, ..., 6, te beschouwen is als een speciaal geval van de visuele discriminatiespanne n, 2n, ..., 6n waarbij alle aantallen bovendrempelig van elkaar onderscheidbaar zijn. De bovengrens van die spanne wordt bepaald door de Weberfractie met een waarde van 1/6. Deze waarde stijgt ver uit boven fracties gevonden voor andersoortige psychofysische variabelen (Ter vergelijking: Wb(lengte)=1/100; Wb(gewicht)=1/35; Wb(helderheid)=1/100).

Door gebruik te maken van zogenaamde gerandomiseerde rangschikkingen werd in hoofdstuk 1 de patroonfactor zoveel mogelijk op de achtergrond gedrukt ten gunste van de factor aantal. In Hoofdstuk 2 waren de rollen precies omgekeerd. Het gaf een formalisering (CODE) te zien van de elementaire Gestaltregel der nabijheid. Er werd van uitgegaan dat een verzameling punten ervaren wordt als een set punten welke elkaar wederzijds beïnvloeden. Deze beïnvloeding liet zich beschrijven middels een binormale dichtheidsverdeling waarbij de spreiding gelijk was aan de helft van de afstand van een punt tot zijn naastbijgelegen buur. Onderlinge beïnvloeding impliceert dan dat relatieve, en niet absolute nabijheid doorslaggevend is. Betrokken op de gehele verzameling punten leidde CODE tot een representatie welke overeenkomstig de visie van Marr (1975) beschouwd kan worden als een beschrijving van een eerste, perceptuele verwerking ('primal sketch'). Criteria, afgeleid uit eenvoudige stippenpatronen (1-punts en 2-puntsfiguur) baanden vervolgens de weg tot een beschrijving van de groepeerbaarheid binnen een willekeurige stippenfiguur in termen van contouren rond (sub-)groepen van stippen.

Aan de hand van een aantal voorbeelden liet Hoofdstuk 2 de kracht van CODE zien. Tekortkomingen kwamen evenwel ook aan het licht hetgeen niet verwonderlijk is. Andere figuur-bepalende regels zoals bijvoorbeeld de regel van de goede continuering en de regel van symmetrie werden in het model niet beschouwd. Verderop komt deze kwestie nog onder de aandacht wanneer invalswegen naar toekomstig onderzoek worden besproken.

De kracht van CODE werd andermaal bevestigd in Hoofdstuk 3 waar, in een drempelexperiment, door CODE voorspelde groepeerbaarheid hoog correleerde met de groepeerbaarheid zoals ervaren door een menselijke waarnemer. Hetzelfde experiment liet tevens zien dat voor de snelle benoeming van het aantal groepjes van stippen soortgelijke beperkingen bestaan als voor de snelle benoeming van het aantal afzonderlijke stippen. Hamilton's hypothese als zou je net zoveel groepjes als eenheden overzien omdat groepjes als eenheden worden beschouwd vindt hiermee bevestiging.

Vervolgens werden door CODE gecontroleerde stippenfiguren gebruikt in een reactietijdsexperiment dat erop was gericht de invloed van figuur-aantal interactie op het telgedrag te onderzoeken. Reactietijden als functie van de groepeerbaarheid der stippen maakten duidelijk dat stippen nooit één-voor-één werden geteld. Op grond van nabijheid goed onderscheidbare groepjes met maximaal 5 stippen werden gesubiteerd, en de deelresultaten ervan steeds toegevoegd aan een lopend totaal. De opteloperatie nam daarbij verreweg de meeste tijd in beslag. Grotere groepen werden op andere grond dan nabijheid opgesplitst in vermoedelijk groepjes van 2 of 3 stippen. De telstrategie was verder overeenkomstig de strategie als bij subiteerbare groepjes gehanteerd hoewel met de beschikbare apparatuur een schatting van subiteertijd en opteltijd niet mogelijk bleek.

Doordat na afsluiting van het in Hoofdstuk 3 gerapporteerde onderzoek geavanceerde apparatuur voor de registratie van oogbewegingen beschikbaar kwam werd een herhalingsonderzoek uitgevoerd waarbij gelijktijdig het oogbewegingsgedrag werd gevolgd. Hoofdstuk 4 deed hier verslag van.

Oogbewegingsdata maakten duidelijk dat een proefpersoon de eerste honderden miliseconden globale informatie van een gepresenteerde stippenfiguur tot zich nam alvorens tot het feitelijke tellen over te gaan. Deze periode van globale perceptie werd gewijd aan het structureren van het visuele veld. Oogbewegingstrajecten in termen van fixatieposities en oogsprongen lieten vervolgens zien dat groepjes stippen bij voorkeur met de klok mee werden gescanned vermoedelijk om verwarring te ontlopen omtrent wat geteld is en wat niet. Een verlenging van de gemiddelde fixatieduur als functie van toenemend aantal stippen verklaarde waarom latenties niet evenredig maar versneld toenamen met aantal. Vroegere verklaringen als zou een positief-versnelde toename het gevolg zijn van een toename in het aantal keren dat opnieuw gestart werd konden door inspectie van de oogbewegingstrajecten worden weerlegd. De verwachting dat oogbewegingsregistraties zouden kunnen laten zien hoe bij grotere groepen stippen segmenteringen worden aangebracht anders dan op basis van nabijheid bleek te optimistisch. Een verklaring voor dit falen werd gevonden in het feit dat een fixatie weliswaar het centrum van het visuele gezichtsveld weergeeft maar niet wat voor preciese informatie er binnen een fixatie wordt ingewonnen.

De registratiemethode bleek erg geschikt om ook oogbewegingen van kinderen te verzamelen. Hoofdstuk 5 deed verslag van een experiment uitgevoerd bij kinderen van rond 5 jaar waarbij gedurende het tellen van kleinertallige stipfiguren het ooggedrag werd gevolgd. De resultaten toonden aan dat zelfs de kleinste aantallen stippen minimaal 4 of 5 fixaties vereisten alvorens een responsie werd gegeven. Een verklaring voor overeenkomstig langere latenties in vergelijking met volwassenen zou gezocht moeten worden in moeilijkheden welke het kind ondervindt bij het ophalen uit het lang-termijnsgeheugen van het bij zo'n stippenaantal behorend label. Het optreden van meerdere fixaties bij de onderhavige aantallen wijst erop dat subiteren in de zin van aantalbenoeming in één oogopslag niet voorkomt bij kinderen rond die leeftijd. Uit de oogbewegingstrajecten behorend bij grotere stipfiguren viel af te lezen dat kinderen, in tegenstelling tot volwassenen, veelal geneigd waren eerst het centrum van de stimulusaanbieding op te zoeken. Dit gedrag werd toegeschreven aan het feit dat kinderen in eerste instantie globale stimuluskenmerken structureren. Vanuit het centrum werd vervolgens naar groepjes van stippen gestuurd in plaats van naar ieder stipje afzonderlijk hetgeen erop wijst dat kinderen gelijk als volwassenen tellen. Daarentegen verklaarde het veelvuldig voorkomen van reïteratief tellen waarom kinderen zoveel langzamer zijn dan volwassenen. De data ondersteunden verder de zienswijze van Fodor (1972) welke erop neerkomt dat kinderen evenals volwassenen over rekensystemen (computational systems) beschikken. Kinderen zijn echter zoveel trager doordat deelprocessen lang niet zo snel en efficiënt verlopen. Met name de dubbeltaak van het opslaan van deelresultaten en het nagenoeg tegelijkertijd verwerken van nieuwe informatie doet waarschijnlijk een al te groot beroep op het concentratievermogen van het kind (zie ook: Vos, in druk).

Aangaande de uitstraling welke dit proefschrift kan hebben op toekomstig onderzoek dient hier eerst vermeld te worden dat per 1-6-83 een ZWO-vervolgproject "Continuering versus Nabijheid" van start is gegaan. In eerste instantie beoogt het project een verrijking van CODE zodanig dat het niet alleen de Gestaltregel der nabijheid maar ook die der goede continuering aankan. Daarbij wordt gedacht aan invoering van een spreidingsfunctie die gevoelig is voor de richtingstrend in een (deel-)set van stippen. Waarnemingsproeven dienen bijvoorbeeld uit te wijzen hoeveel stippen noodzakelijk moeten samenwerken om de regel der goede continuering doorslaggevend te laten zijn. Naast parameterschattingen voortvloeiend uit dergelijke proeven kunnen handig gekozen stipfiguren wellicht het nodige leren. Zeven of meer stippen gelegen op de hoekpunten van een regelmatige veelhoek worden veeleer waargenomen als een cirkel dan als een veelhoek. Het aantal stippen gelegen in een spiraalvorm alsmede de uitdijing van die vorm lijken onderhevig aan minimale criteria opdat de verzameling stippen nog als spiraalvormig wordt herkend. Dergelijke figuren lijken geschikt om te worden ingezet bij copieerproeven waarin kinderen de vraag wordt gesteld om stippen met elkaar te verbinden die naar zijn/haar mening bij elkaar horen (of figuurtjes vormen). Het uittesten en afnemen van deshabituatieproeven met baby's is ook opgenomen in de planning van het vervolgonderzoek. Die proeven moeten ons laten zien of de waarneming al dan niet van meet af aan in even grote mate gehoorzaamt aan de twee elementaire regels, -nabijheid en continuering. Habituatie is vermoedelijk het geval wanneer twee half over elkaar liggende, cirkelvormige stipfiguren langzaam uit elkaar getrokken worden tot ze naast elkaar liggen. Deshabituatie wanneer een van beide uiteindelijk maanvormig blijkt te zijn.

Het is bekend dat regelmatige stipfiguren in het algemeen overschat worden terwijl onregelmatige zoals gerandomiseerde stippenpatronen worden onderschat. Recente loodsproeven doen vermoeden dat over- en onderschatting mede het gevolg zijn van relatieve oppervlakteschatting. Rekening houdend met het feit dat CODE naast het omringende contour ook het (relatieve) oppervlak van een (deel-)verzameling stippen kan berekenen dient zich hier een andere weg naar toekomstig onderzoek aan: de invloed van figuur-aantal interactie op het schatten van grote verzamelingen.

Dit proefschrift betrof thematiek op het gebied van de visuele waarneming en daarmee samenhangende cognitieve processen. Alle gerapporteerde onderzoek had betrekking op waarneming en verwerking van de meest eenvoudige visuele figuren, opgebouwd uit identieke stippen. Het gebruik van deze figuren heeft in niet geringe mate bijgedragen tot vergemakkelijking van de proefondervindelijke beantwoording der gestelde vragen. De verwachting lijkt gewettigd dat ook in toekomstig aansluitend onderzoek naar de rol van andersoortige Gestaltregels bij waarneming en cognitie dezelfde soort stipfiguren hun puntigheid opnieuw zullen bewijzen.

Referenties.

- Averbach, E. The span of apprehension as a function of exposure duration. Journal of Verbal Learning and Verbal Behavior, 1963, 2, 60-64.

- Fodor, W. Some reflections on L.S. Vygotky's "Thought and Language". Cognition, 1972, 1, 83-95.

- Jevons, W.S. The power of numerical discrimination. Nature, 1871, 3, 281-282.

- Marr, D. Analyzing natural images, a computational theory of vision. M.I.T. AI Lab, AI memo 334, 1975.

- Vos, P.G. De rol van het geheugen in hoofdrekenen. In Vos, P.G., Koster, N., en Kingma, J. (Eds.), Rekenen. Hillegom: Swets en Zeitlinger (in druk).

Summary.

This thesis concerns the perception and processing of visual number as it is represented in the form of numbers of dots in dot figures. In the introductary of the thesis it is remarked that the problemfield resulted from an originally philosophical interest in human attention and its restrictions. Gradually this interest shifted to experimental research aimed to model number-naming strategies (subitizing, counting and estimating) used by people when ascertaining the number of visually presented dots. This research, however, remained quite uncertain about the very processes underlying such strategies.

In chapter 1 a probabilistic model is presented for the immediate apprehension of visual number. The model implies the hypothesis that subitizing is a consequence of supra-liminal discrimination between internal represenations of small neighbouring numbers. Chapter 1 further reports a reaction-time experiment and a threshold experiment in which the validity of the model is tested. The experimental results supported the hypothesis mentioned.

Not only the number of dots also the way they are arranged into configurations plays an important role in number-naming. The Gestalt rule of proximity is an important guideline to understand such configurations. Chapter 2, therefore, presents a model (CODE) which formalizes the Gestalt rule of (relative) proximity. Its validity towards the perception of dot figures and its usefulness towards a number of problemfields are further discussed.

The first part of chapter 3 is a report of a threshold experiment in which CODE-controled dot figures were presented. The experimental results supported CODE. Chapter 3 further presents a model which describes the influence of the number-pattern interaction on counting large-numbered dot figures. Next, the

predictability of the model is tested in a reaction-time experiment with regard to latencies as a function of both the number of dots and their groupability. It was found that dot figures are preferably counted in groups: small proximity-based groups of at most five dots are subitized while the partial results are added to a running total; larger groups of dots are subdivided on the basis of cues other than proximity. Their partial results are also added to a running total. It is found that the operation of adding the partial results is the major contribuent to overall latency.

Chapter 4 reports a similar reaction-time experiment as reported in chapter 3 but now eye movements are also measured. Spatial and temporal information from the eye movement data are giving further support to the hypothesis that dot figures are counted by groups.

The question of how children around the age of five count dot figures is the central theme of chapter 5. Children's eye movement information from a simple dot enumeration task shows that children employ similar counting strategies as are used by adults; their efficiency, speed and range, however, are much smaller than those reported for adults.

The epilogue resumes the major conclusions and puts foreward some lines for further research: an enrichment of CODE such that it also can handle the Gestalt rule of good continuation, and the investigation of under- and overestimation of arrangements of dots in relation to spatial illusions.

CURRICULUM VITAE

Michaël Pieter Johannes van Oeffelen werd geboren op 6 mei 1952 te Breda. Na het behalen van het HBS-B diploma in 1970 begon hij zijn studie Natuurkunde (N4) aan de Universiteit van Nijmegen. In oktober 1974 legde hij het kandidaatsexamen Natuurkunde af met Scheikunde als tweede hoofdvak. Het hoofdvak Psychofysica van de doktoraalstudie Natuurkunde werd verricht onder leiding van Harm Jongsma op het Laboratorium voor Medische Fysica en Biofysica te Nijmegen en betrof de ontwikkeling van psychofysische methoden voor onderzoek naar visuele vormperceptie. Het doktoraalexamen werd afgelegd in maart 1979. Van mei 1979 tot mei 1982 was hij als wetenschappelijk medewerker in dienst van de Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek, en was werkzaam bij de vakgroep Ontwikkelingspsychologie te Nijmegen op het project ´waarnemen en benoemen van visueel aantal´ (ZWO-projectnr. 15-32-05). De laatste drie maanden van 1982 was hij als wetenschappelijk medewerker in dienst van de vakgroep Ontwikkelingspsychologie waar hij als leider fungeerde van het tweede-jaars praktikum Ontwikkelingspsychologie.

STELLINGEN

1 The classical process of subitizing is nothing but a special case of (number-) discrimination.

(This thesis).

2 Intuitive Gestalttheoretical concepts such as the rule of grouping elements on the basis of proximity can be described by psychologically plausible formal models.

(This thesis).

3 Generally, the field of attention does not match the field of foveal vision; hence, eye movement registration is not suited for the analysis of cognitive processes in vivo.

(This thesis).

4 Shannon's (1951) guessing game method is well adapted to the analysis of human inductive reasoning processes involved e.g. in the extrapolation of incompleted letter patterns.

(Vos, P.G. & van Oeffelen, M.P. Strategies for the extrapolation of letter series. Internal report: 830N02, 1983, Nijmegen.)

(Shannon, C.E. The prediction and entropy of printed English, Bell Systems Technical Journal, 1951, 30, 50-64.)

5 When the result of an A-reaction has subsequently to be used in the realisation of a second reaction, the initial A-reaction is systematically lengthened.

(Vos, P.G. & van Oeffelen, M.P. Kwantificeren van hoeveelheden: chronometrische analyse. In J.G. Lodewijks & P.R. Simons (Ed.), Strategieën in leren en ontwikkeling. Lisse: Swets & Zeitlinger, 1982.)

6 Soviet-russian learning approaches towards the child's mathematical development are not incompatible with Thorndike's associationistic approach of learning processes.

7 The scientific progress in developmental psychology can best be assured by an organizational dichotomy into interaction-oriented and process-oriented working-units.

8 Complex numbers should be introduced much earlier in high-school mathematics education than it is nowadays the rule.

9 It is easier to program people than computers.

10 De verrichtingen op voetbalvelden laten herhaaldelijk zien dat het geheel (het elftal) minder kan zijn dan de som zijner delen (de spelers).

Deze stellingen behoren bij het proefschrift:

'The perception and processing of visual numerosity'

M.P.J. van Oeffelen.

Katholieke Universiteit Nijmegen, 9 februari 1984, 16.00 uur.